# Texas Mathematics Teacher

**Activities and Games for Decimal Notation**

**Why Manipulatives Work**

**The Radial Plane-- A Model of Spherical Geometry**

**Focus on Factoring**

## OCTOBER 1993

**TEXAS MATHEMATICS TEACHER**, the official journal of the Texas Council of Teachers of Mathematics, is published four times each year, in October, January, March, and May. All manuscripts and editorial correspondence should be addressed to George H. Willson, listed above.

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# TEXAS MATHEMATICS TEACHER

TEXAS COUNCIL OF TEACHERS OF MATHEMATICS

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TEXAS MATHEMATICS TEACHER
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## President's Message, TCTM - Year in Review

As I think back over my past year as TCTM President, I cannot help but be impressed by the many members who put in countless hours to help the organization have a successful year. Read on to learn what your $8 dues has helped us accomplish!!

Communication with members is a primary goal of our organization. This past year you should have received four *Texas Mathematics Teacher* journals, and two *Math Talk* newsletters with the *STEAM Across Texas* publication enclosed. George Willson is responsible for the journal and Dorothy Ware edits the STEAM publication. Those are huge jobs, and I appreciate their hard work. If you would like to help in either one of those projects, let George or Dorothy know. If you are not receiving these publications regularly, we want to know about it. The mailing label on this journal indicates when your dues expire - be sure to stay current so that you won't miss out on these mailings. Members are also sent the CAMT program booklet each spring.

One of the major functions of the Texas Council of Teachers of Mathematics is to co-sponsor the annual Conference for the Advancement of Mathematics Teaching. Other sponsors are the Texas Education Agency, the Texas Section of the Mathematical Association of America, and the Texas Association of Supervisors of Mathematics. TCTM is in charge of on-site registration, and many of our members worked at the registration desk during the conference. Thanks to your hard work, lines were short or non-existent. We also are in charge of NCTM material sales, and Cindy Schaefer did a wonderful job of organizing that task. The TCTM breakfast meeting was held during CAMT and those in attendance appreciated the hard work of Frances Thompson in making the arrangements and obtaining the MANY door prized that were given away! Next summer CAMT will be held July 27 - 29 at the George R. Brown Convention Center in Houston.

At the CAMT luncheon, the 1993 Texas teachers selected as state-level Presidential Awardees in Mathematics were honored. TCTM is please to have been able to give each of these outstanding teachers $50 to help cover their CAMT expenses.

One of the most enjoyable tasks I have as your president is getting to read the college scholarship applications from high school seniors. With help again from a generous contribution from Prentice Hall, TCTM awarded a total of four thousand dollars in scholarship money to six student who plan to pursue careers in mathematics education. Many outstanding young people applied, and the scholarship committee had a difficult job making their selections. Look for the 1994 scholarship application form in the spring *Math Talk* newsletter.

Next summer we will be awarding at least six $100 CAMTerships to teachers who are TCTM members and have just completed their first year in the classroom. We hope that this will encourage new teachers to attend CAMT, and help them with the expenses they incur in doing so. More information about this project will be in the fall *Math Talk.*

If you have ideas about ways we can improve TCTM, or if you want to become more involved, I want to hear from you! I challenge you to become involved in your professional organizations, and be a positive voice for mathematics education in your school and in your community. Have a good year.

Susan Thomas
TCTM President

# ACTIVITIES AND GAMES FOR DECIMAL NOTATION

**Mona Edwards**

*Owasso, Oklahoma*

**Jean Bosch**

*Richmond, Texas*

OBJECTIVE: Junior High School students will be able to correctly write decimal numerals which are dictated or shown to them in word form.

Activity 1:

Magically change the decimal point into a basketball and you will simultaneously infuse your class with enthusiasm and guaranteed fun and success! Take them to the gym, and you'll be able to feel their excitement. That's exactly what you get to do with this activity.

Before going to the gym with the class, you will need to prepare two large butcher paper strips with about six lines drawn to indicate place value. The strips should be posted horizontally beside each other on a wall and should look like this:

You will need a box containing the numbers 0-9 and a decimal point, which have been written on individual index cards. (You'll have a total of

11 cards including the decimal point card.) Also write two sets of the numbers 0-9 in large print on typing paper.

Take the class to the gym and divide them into two teams. Ideally, there should be eleven students on each team so that each student can hold one item. Give each team a set of the typing paper numbers and a basketball (alias, the basketball point). The students should distribute the typing paper numbers and basketball among them so that each student has one item. A team captain or another student can be in charge of the basketball or "decimal point." Line each team up at least twenty feet in front of their paper strip and parallel to it.

Explain to them that you will draw numbers out of the box to form a decimal numeral (which they will not get to see), and that you will then dictate the number name to them. For example, suppose you draw in order a 2, a 6, a decimal point, a 5, a 7, and then a 1. You will then dictate "twenty-six and five hundred seventy-one thousandths."

As soon as the number name is dictated, the team members should rush to their correct positions in front of the butcher paper strips, with the left-most position on the paper being assigned to the largest place value of the number being formed. When students are in place, they should hold their typing paper numbers up in front of them. The first team to correctly get in place gets the point. Note that all team members will not play each round because all numbers will not be called, but the decimal point should always be used, whether or not the decimal card is drawn, to stress that the decimal point is understood to be to the right of the units place of all whole numbers.

Note that you should draw as many or as few cards as your class is ready for, eventually making it harder. You may or may not draw the decimal point card each time, and, of course, if the 0 card is drawn first, the teacher will disregard it in the dictation.

The students love this and become very clever. They learn to exchange numbers on their team frequently so that any members from the other team who do not yet understand place value cannot watch someone on their team who might have the same number they do and just see where they are supposed to go. Conversely, while each team is intent on not helping the other team, the teams do talk with and help their own members.

A variation of this game uses only one strip and lets the teams take turns. A time limit of one minute is given on either game after the first day.

In later games the number can be written out in words on the overhead rather than dictated orally. The game continues as before.

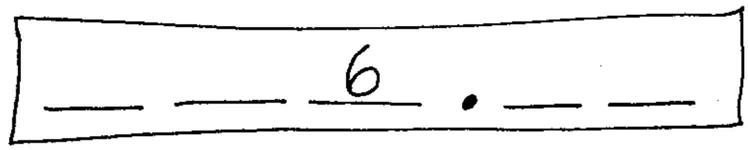
Activity 2:

A follow-up activity which unobtrusively brings students closer to pencil, paper, and decimal points involves prepared sentence strips and dried pinto or red beans. The teacher gives each student or set of partners a sentence strip that is prepared like this:

___ ___ ___ ___ . ___ ___

They also get half of an index card with a one digit number written on it. For ease of checking, they should all have the same number. They will all need a bean which will represent the decimal point. The bean is really not necessary since the decimal point is written on the sentence strip, but it adds interest, stress, and physical dimension to the activity. The students just put the bean over the decimal point on their sentence strip.

The teacher will then dictate a number that should include the number that is written on the index cards. If the students' index cards have a 6 written on them, for example, there will be a 6 in all dictated numbers. Then the teacher might dictate the number "one hundred six and thirty-four hundredths." The students should write down what they think the teacher has said on a scratch paper and then put the 6 and the bean where they should go on the sentence strip. In this case, the students' strips should look like as follows:




It is very easy to check on the progress of your students with a glance around the room.

This activity, too, can be modified by giving each set of partners the full set of halved index cards with numbers 0-9 written on them, a bean, and a sentence strip without the decimal point written in a permanent position.

These activities provide a kinesthetic learning mode by which to drill, increase student interest, and provide ease in immediate monitoring of student progress. They also allow for cooperative learning and are enjoyable because of the success students experience and the joy they have in the process of learning something they will never forget. Enjoy!

# WHY MANIPULATIVES WORK

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**Hsuan Kung-Shyu**

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Hear.... and, Forget
See.... and, Remember
Do.... and, Understand.
--Old Chinese Proverb

If you don't have a dog,
you go hunting with the cat.
--Old Brazilian Proverb

Proverbial Comments

Popular wisdom often not only scoops the slower-paced, though more sure-footed, results of scientific investigation, but it also encapsulates these results in a more palatable manner. Thus, the Chinese proverb cited above summarizes, in a form that virtually commands immediate assent, the basic findings on the use of manipulative materials in the teaching of elementary mathematics. Fennema (1972) and Sowell (1989), for example, reviewed the research on the effectiveness of mathematics lessons which included the use of manipulatives. Both concluded that mathematics achievement is substantially increased by the long-term use of concrete instructional materials; student's attitudes toward this too often dreaded subject were also improved by the incorporation of manipulatives into mathematics lessons.

Manipulatives, however, are not a panacea; they promote, but do not insure success. Indeed, a well planned and nicely executed lesson using appropriate manipulative materials may leave some children cold. Hence, the teacher must possess the resourcefulness that the above cited Brazilian proverb admonishes of us. Different instructional approaches for presenting the same content may be necessary in order to reach the whole class. Moreover, should the "dog" turn out to be a sophisticated set of industrialized materials, which may just not be available (or not available in sufficient quantities) in many school systems, the "cat" can always be homemade materials or materials that the students themselves collect and/or assemble.

Nevertheless, the creative, caring teacher may not be getting the most out of the manipulatives used in the classroom. Specific lesson guidelines for using the materials at hand are not enough. Indeed, we propose that one of the most important aspects of the use of manipulatives is the teacher's creative input--and by "input" we here include such notions as the teacher's knowing when and how to cue the students and when to back off and leave the students to their own devices. The quality of this teacher input, however, depends, at least partially, on the teacher's understanding of why manipulatives work since this understanding promotes a sympathetic, almost mystical, three-cornered relation between the teacher, the student, and the materials used. (The more prosaic question of how manipulatives work is, of course, addressed in most lesson guides.) Unfortunately, there is no one definitive answer to this question. In what follows, therefore, we will try to address the question of why manipulatives work from various theoretical viewpoints--so many "cats," as it were, in our hunt for ever more effective teaching.

## The CIP Approach

One important approach to human learning current among education specialists is the Cognitive Information Processing (CIP) model. The fundamental metaphor guiding research by CIP theorists is that the human mind works pretty much like a sophisticated electronic computer. This framework is, of course, not surprising since one of the basic motivations for the development of "thinking machines" was to design a mechanical model of human thought. Although it is not a very suggestive model for the important affective and volitional aspects of human experience, the CIP model has proved to be very fruitful in the investigation of the cognitive domain.

According to the CIP model, when sets of stimuli are received by the human organism, some of them are filtered into the short-term (or working) memory by devices that direct and maintain our attention. Selected stimuli are then encoded for transferal to the long-term memory for storage and eventual retrieval. The model can be elaborated in various ways, such as the inclusion of a set of master controls which selects appropriate strategies for a given task; indeed, further elaborations may be expected as our engineers discover ever more wrinkles in their electronic grey matter. Nevertheless, this short summary will suffice for our purpose here.

The CIP model posits certain internal processes that may be influenced by external events (Gagné, 1985). Among the former (internal processes) are such activities as the aforementioned encoding and storage of cognitive material. Although the means by which the nervous system effects these processes are not completely known, CIP based research seems to have succeeded in identifying various external events that do promote learning. Among these external events, we may classify instruction with manipulative materials. Manipulatives help to direct and focus attention by providing concrete instances with which to work. Thus, in developing the concept of fractions, the use of fraction rods, for example, provide specific tasks for the learners, thereby helping to focus and maintain their attention.

Being physical embodiments of concepts, manipulatives also increase the distinctiveness of the concepts being studied, which in turn facilitates both the ability to discriminate a given concept from others and the ability to encode and store the concept in an easily retrievable manner. "One third," for example, becomes more distinct for the student when it is actually marked off from a unit and seen to be a determinant part of that unit; it also becomes possible to directly compare thirds and, say, fourths, which helps the student to develop intuitions about these concepts. By multiplying the number of senses involved in the learning situation, manipulatives increase the depth of processing which, again, promotes good encoding. Indeed, depth of processing is an important notion, but, since it depends on the idea of mental schemata which the CIP model took over from the developmentalists, we will defer further discussion of it for the moment.

We often hear the truism that the teacher can teach, but only the student can do the learning. The CIP model makes this truism more precise in that it takes into account both the internal processes and external events in any learning situation. Manipulatives aid both aspects. They help the student to become actively involved with the material that is to be learned and, thus, trigger the internal cognitive processes of the student. They also help the teacher to influence the learning process by setting up those external events that foster the effective execution of the internal processes.

## The Developmental Approach

Piaget advances the doctrine that children develop according to fixed stages at more or less fixed ages. Were it possible to correlate these developmental stages, marked by emergent mental capabilities, with the maturation of various structures of the cerebro-nervous system, the doctrine would be of great interest. Not only has no such correlation been demonstrated, however, but psychological research has found virtually no support for the concept of fixed developmental stages and, thus, this aspect of Piaget's theory has not found much acceptance in the scientific community. In his later work, even Piaget himself downplayed the idea of fixed stages.

Other aspects of Piaget's approach, however, have been of enormous influence. Thus, Piaget identified two different types of knowledge: physical knowledge of objects in external reality and logico-mathematical knowledge of relationships among these objects. Seeing a pile of buttons is an example of physical knowledge and even partitioning that pile into three (equinumerous) smaller piles does not in itself go much beyond physical knowledge. When, however, the three piles are recognized to be the same (that is, equinumerous) and are related to the original pile as determinant parts, then logico-mathematical knowledge emerges since the learner is no longer concentrating on the piles of buttons themselves, but on their relationships to one another and to the original pile. This distinction has evolved into the idea of different levels of abstraction, each level being more or less dependent on preceding levels. The idea here is not one of emergent developmental stages in the learner, but rather one of a structure inherent in the material to be learned.

In Piagetian terms, therefore, manipulatives would be important devices for acquiring both types of knowledge. We acquire physical knowledge, of course, through the senses. The acquisition of logico-mathematical knowledge is also promoted by the use of manipulatives, however, because the relationships among objects that constitute this type of knowledge are developed by observing objects in interaction. Thus, when the above mentioned pile of buttons is linked to the partition by the appropriate concepts, the idea of fractions may be developed. Manipulatives provide the occasion for the child to observe how objects relate to each other and how they react to the child's own actions.

Another fruitful aspect of Piaget's thought is that logico-mathematical knowledge is not just an internal replica of external relationships lifted, somehow, form the object-world; rather, it is a mental creation of the relationships involved. Piagetian research demonstrated a constructivist principle according to which children build their knowledge in interaction with the environment through their own mental activity. Thus, manipulatives--and here we must insist on the correct meaning of the word, materials that the children actually manipulate themselves--manipulatives are the proving ground for the child's mental activity. By experimenting with various relations among the manipulative materials (and, of course, with judicious cuing and feedback from the teacher), the child gradually builds up coherent patterns of relationships. Thus, it is not sufficient to give a child a pile of beads and expect the development of the whole system of fractions in short order. Rather, various mediating concepts need to be developed. The teacher can foster this development by setting the student structured tasks, encouraging the student to talk about emerging concepts, and testing various ideas using the materials at hand.

## Schematizers

Perhaps the most fruitful Piagetian concept is that of a schema. The notion is virtually ubiquitous in contemporary theorizing about the learning process. Simply put, a schema is a pattern of relationships, or a conceptual structure, built up by the knower. Understanding a concept consists in fitting it into a schema; that is, putting it in its right place in relation to other concepts. Thus, schemata are not static edifices, but dynamic structures that may be constantly elaborated and otherwise modified or even abandoned for other schemata.

On the one hand, each schema is composed of bits of knowledge (though this is a simplification), but on the other hand the schemata that we have already built up effect the processes of further knowledge acquisition since the new knowledge must be fitted into a schema. Each schema may be more or less elaborated and each bit of knowledge may be placed in a more than one schema. The more that each schema is filled out and the more connections there are between various schemata, the more is the depth of processing (on the CIP model) increased and, consequently, both understanding and recall improved. Thus, the concept of division is a complex schema which includes the subsidiary concepts of grouping and sharing, while the concept of fractions is also a complex schema which includes such notions as (equal) parts. But both sharing and grouping are also applicable to the concept of fraction and, thus, by relating the new schema of fractions to the already developed schema of division, a richer and more meaningful conception of fractions emerges.

We have already indicated that manipulatives enhance that mental activity that results in the construction of conceptual structures. They also promote greater understanding by fostering greater elaboration of each schema and a larger number of connections between schemata. There may be still another way in which manipulatives promote instruction, however, for schemata cannot themselves be transferred directly from the teacher to the child (Skemp, 1987). By presenting the child with structured (from the teacher's viewpoint) manipulative materials, the child is more likely to construct structures corresponding to the concepts that the teacher is trying to impart.

## In Lieu of a Conclusion

Since the purpose of this paper is to address, from different theoretical positions, the question of why manipulatives work, a traditional conclusion to the paper does not seem entirely appropriate. In lieu of such a conclusion, therefore, we offer a few words on two additional topics: the NCTM's *Standards* and the effects that manipulatives may have on the affective and volitional aspects of the learning situation.

The NCTM's *Standards* calls for a reconceptualization of the way that mathematics is taught. In particular, it is suggested that learners not be treated as passive recipients in the acquisition of merely computational skills by rote learning, but rather as active participants in the building up of their own mathematical knowledge. Regardless of the particular content being taught, manipulatives address this fundamental expectation by encouraging children to investigate the mathematical properties under discussion, as well as fostering the formulation and testing of mathematical concepts by the children themselves. The *Standards* explicitly recognize the effectiveness of manipulatives, moreover, by recommending that the classroom be stocked with appropriate learning materials, including simple household supplies like beans and buttons.

Finally, we would like to comment on the affective and volitional aspects of manipulatives. It is perhaps patent that the child's affective relation to mathematics will be at least partially determined by the child's ability or inability to understand the mathematical concepts being taught and to successfully complete the tasks assigned by the teacher. Since manipulatives promote understanding, they may also be expected to promote positive attitudes. The external approbation of success or frustration of failure, however, are but a small part of the affective relations involved. Indeed, we believe that the human activity of creation is an inherently pleasurable activity and thus the role of manipulatives in liberating the child's creative forces in the construction of mathematical structures is of the utmost importance in developing positive affective attitudes towards mathematics.

The volitional aspects of learning are almost always forgotten in discussions about learning processes. Nevertheless, to use Plato's evocative phrase, "the spirited part of the soul" may determine the quality of any given learning experience. This is because learning is, as we have already seen, an activity undertaken by the individual and because the level of activity undertaken by any individual is determined by that individual's volition. As a first approximation, we may distinguish the following four points in the continuum of activity levels: the individual may wholeheartedly undertake a given project; halfheartedly go along with the teacher; halfheartedly resist the teacher; or wholeheartedly oppose a given project. Thus, the individual makes a conscious decision--or, at any rate, a decision that can be made conscious upon reflection--to undertake a given project at some activity level. Since the individual's affective relation to the project to be undertaken is a major influence in the decision, manipulatives may inspire higher activity levels by promoting positive affective relations towards mathematics.*

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# THE RADIAL PLANE— A MODEL OF SPHERICAL GEOMETRY

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The purpose of this brief note is to introduce a very intriguing, yet simple, model of a spherical (non-Euclidean) geometry and some of its properties.

This model may be used as a source of challenging enrichment activities for geometry students. The axioms, theorems, and the geometric objects in this model are distinctly different from the Euclidean plane, which may have become "obvious" to them. For example, consider the $\Delta$ABC in Figure 4, (page 20). The development of the examples, which can be easily sketched in the plane, will require the students to carefully reconsider the definitions of the geometric objects and the meaning and implications of the axioms, which they may have begun to overlook. It provides an uncluttered study of axiomatic systems and how the negation of a few key axioms can have large effects on their consequences, the theorems.

I recommend that the points, lines, distance, and angular measure should be defined and a number of examples such as: line, ray, angle, and segments, should be presented. The student, or study group, will benefit by discovering the various examples contained in this note. The other concepts, like convexity and the plane separation axiom, can be introduced as the study evolves.

All terms in this note are consistent with the text by E. E. Moise. In section one, the Radial Plane geometry is defined, and in section two, many of the interesting properties of this model are presented. In section three, a comparison of this geometry with the geometry of the sphere is provided, as well as an example of the Radial Plane geometry in the Minkowski Light cone. The discovery that this model is the natural geometry generated by cones has developed as a pleasant bonus in this study.

1. Description of the Radial Plane Geometry.

The *Radial Plane Geometry* is defined as follows: Let the space be the points of the Cartesian plane. A line will be the union of any two distinct rays with the origin, 0, as the common endpoint. If the rays are collinear, then they are also opposite rays. As illustrated in Figure 1, the plane is partitioned into three disjoint sets, line $\overline{A}\,\overline{B}$ and the halfplanes $H_1$ and $H_2$.

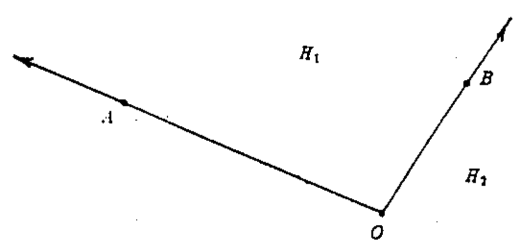



Figure 1.

All lines intersect at the origin, and thus, there are no parallel lines (spherical geometry). If P and Q are on the same ray with its endpoint at 0, then the distance from P to Q, d(P,Q), is the usual Pythagorean distance from P to Q, $d_U(P,Q)$. Observe that there are infinitely many lines containing P and Q, when P and Q are on the same ray with its endpoint at the origin. Thus, two points do not determine a unique line, and thus this incidence axiom fails in the Radial Plane, as it does in the geometry of the sphere. If P and Q are on different rays with endpoints at 0, then $d(P,Q)$ - $d_U(P,0)$ + $d_U(0,Q)$. (d is the well-known radial metric for the plane.) Unlike the is, all lines have coordinate systems, all distance properties hold, and all betweenness properties are valid. The angular measure m for this geometry uses the usual angular (protractor) measure $m_U$ from the Euclidean Plane, where $0 < m_U \angle A < 180°$, for any $\angle A$ and the usual interior of an angle, $Int_U$. Since angles are the union of noncollinear rays, with the same endpoint, no angle has the origin as its vertex. If A ($\neq 0$) is the vertex of an angle, then since the rays of an angle must be noncollinear, the rays can not be opposite rays with endpoint at A. In fact, an angle must be of the form $\angle BAC = \overline{AB} \cup \overline{AC}$, where 0 is between A and B and 0 is between A and C as shown in Figure 2.

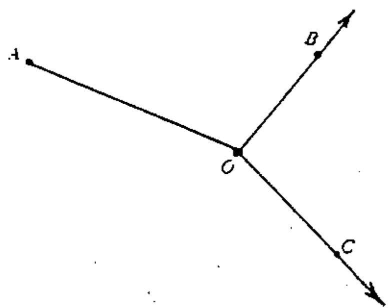
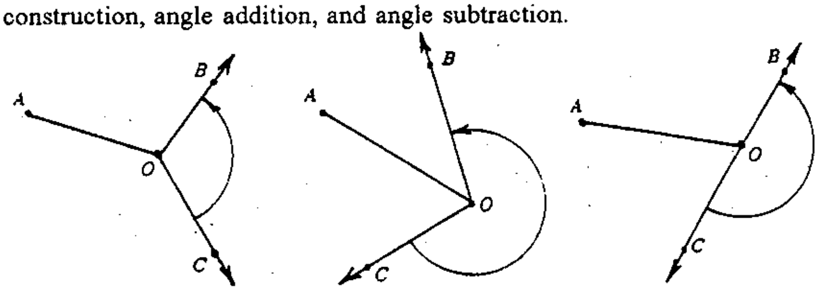


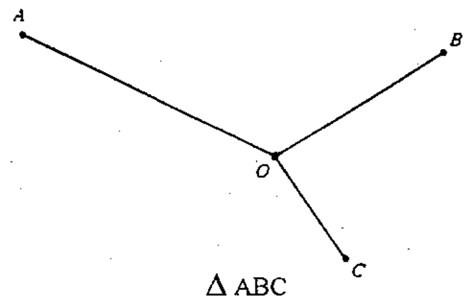

Figure 2.

We define, m $\angle BAC$ - $m_U \angle BOC$, whenever $A \notin Int_U \angle BOC$; $m\angle BAC$ - $360°$ - $m_U \angle BOC$, whenever $A \in Int_U \angle BOC$; and finally $m \angle BAC = 180°$, when $m_U \angle BOC = 180°$ as shown in Figure 3 a,b,c. All of the expected properties of angular measure are valid for m, such as the axioms for angle construction, angle addition, and angle subtraction.




a. b. c.

Figure 3.

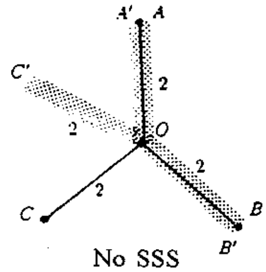



Δ ABC
Figure 4.

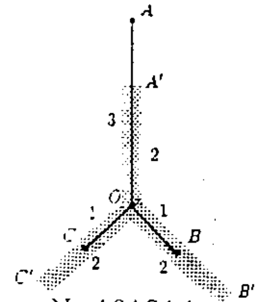
2. Properties of the Radial Plane Geometry.

Many geometric axioms are satisfied in the Radial Plane, such as the Ruler Axiom. (i.e. Every line has a distance preserving coordinate system.) Also, all betweenness properties and angular measure properties are valid. This geometric model serves as a powerfully instructive example of how the failure of one incidence axiom and the Plane Separation Axiom cause practically all of the standard geometric properties to fail. Several of the standard properties which fail in the Radial Plane are exhibited in the following presentation. While two points on the same ray with endpoints at the origin are contained in infinitely many different lines, two points which are not on the same ray with endpoint at the origin are contained in a unique line and every pair of points determines a unique interval. It is clear, as in Figure 4, that the sum of the measures of the angles in any triangle is 360°. Triangles do not have exterior angles. There are no trapezoids or parallelograms, because there are no parallel lines.

The triangle congruence and similarity properties are rather interesting. In Figure 5, Δ ABC corresponds to Δ A'B'C' in such a way that side-side-side are congruent to the corresponding side-side-side and Δ ABC is not congruent to Δ A'B'C'. Regarding other congruence properties, observe in Figure 6 angle-side-angle-side-angle-angle in ΔABC are congruent to the corresponding angle-side-angle-side-angle-angle in Δ A'B'C', but Δ ABC is not congruent to Δ A'B'C'. Observe that the case ASASAA, includes each of the cases ASA, SAS, SAA, SSA, and AAA. Thus, no congruence theorems are true for the Radial Plane.

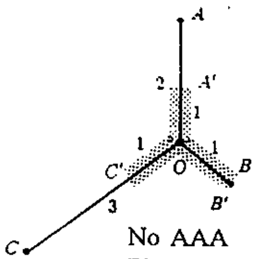



No SSS
Figure 5.

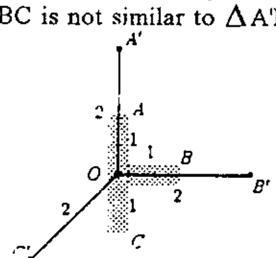



No ASASAA
Figure 6.

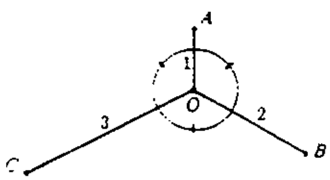
In Figure 7, $\Delta$ABC corresponds to $\Delta$A'B'C' in such a way that angle-angle-angle are congruent to the corresponding angle-angle-angle, but $\Delta$ABC is not similar to $\Delta$ A'B'C' and in Figure 8, side-angle-side-side-side correspond in such a way that corresponding angles are congruent and corresponding sides are proportional, but $\Delta$ABC is not similar to $\Delta$A'B'C'.



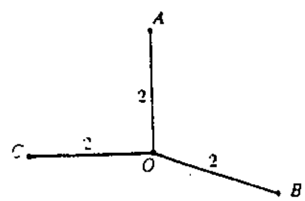

No AAA
Figure 7.




No SASSS
Figure 8.

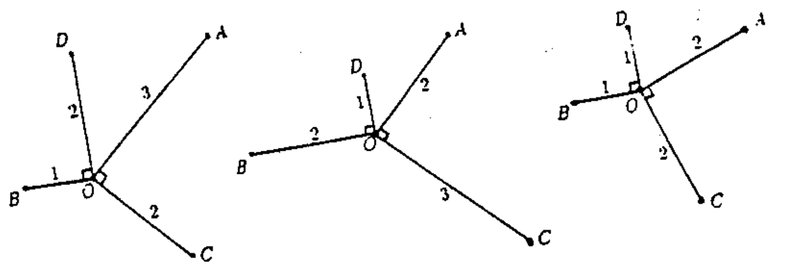



equiangular does not imply equilateral
Figure 9.




equilateral does not imply equiangular
Figure 10.

Recall that a quadrilateral □ABCD is a *Saccheri Quadrilateral* provided: m∠B = m∠C = 90° and AB = CD. Saccheri proposed that quadrilaterals such as this could be proven, without using Euclid's Parallel Axiom, to be rectangles. He proposed that this would show that this parallel axiom could be proven from the previous axioms. In this geometry, with no parallel lines, Saccheri quadrilaterals are rectangles, because the upper base angles ∠BAD and ∠ADC are also right angles, the diagonals, $\overline{DB}$ and $\overline{AC}$, need not be congruent, and the upper base $\overline{AD}$ may be longer, or shorter, or congruent to the lower base $\overline{BC}$. These possibilities for Saccheri quadrilaterals □ABCD are illustrated in Figure 11a,b,c.

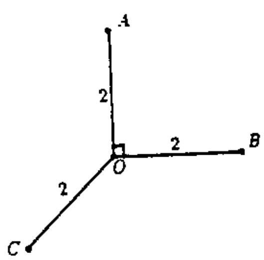



BC = 3, AD = 5
a.

BC = 5, AD = 3
b.

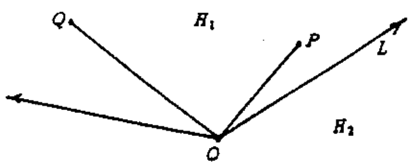
BC = 3, AD = 3
c.

Figure 11.




The Pythagorean Theorem fails.
Figure 12.

Recall that a *convex* set S has the property that if P and Q are any two points in S, then $\overline{PQ} \subset S$. The Plane Separation Axiom [1] can be stated as: if L is a line in a plane E then 1.) E-L is the union of two disjoint convex sets $H_1$ and $H_2$ and 2.) if $P \in H_1$ and $Q \in H_2$, then $\overline{PQ} \cap L \neq \emptyset$. The Radial Plane fails to satisfy this postulate, because $H_1$ and $H_2$ are not convex as seen in Figure 13.




$\overline{PQ} \not\subset H_1$

Figure 13.

In [1] the side-angle-side axiom is used to prove the triangle inequality. In this geometry we see that the distance function d satisfies the triangle inequality and this does not imply the side-angle-side congruence property. Recall that a circle with center at point A and radius r is the set of all points in a plane containing A, which are r distance from A. Consider, in Figure 14a, the intersection of the circle $C_1$ with center at (1,0) and radius 3 with the circle $C_2$ with center at (-1,0) and radius 3 (different circles that intersect in infinitely many points) or, in Figure 14b, the intersection of $C_1$ with the circle $C_3$ with center at the origin (0,0) and radius 3 (nonintersecting circles of radius 3 with distance between their centers equal to 1). The equations which determine the coordinate values of the points on the graphs of conic sections in the Euclidean plane are not valid in this geometry, because of the inadequacy of the Pythagorean theorem to determine distances. However, by using the definitions of the conic sections, the reader may find the determination of the graphs of ellipses, hyperbolas, and parabolas will provide a good analytic geometry exercise in the application of the non-algebraic definitions of the conic sections. Here we mean the definition in terms of constant sums (ellipses), constant differences (hyperbolas), and equidistant from a point to a line (parabolas).

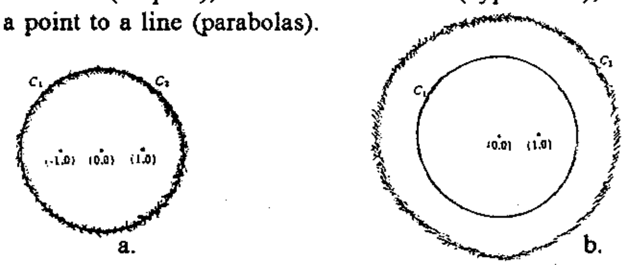



Figure 14.

3. A Comparison and an Example.

The radial plane geometry may seem unnatural and not very "geometric." However, its properties are at least as natural as, and may be compared with, the geometry of the earth (sphere). In fact, these two geometries are similarly generated from intersections of flat planes with common surfaces in three-space. The lines in the geometry of the sphere are generated as the intersection of the sphere and (flat) planes passing through the center of the sphere. Similarly, every line in the radial plane is the intersection of a cone and a (flat) plane containing its apex and a point in its interior, as in Figure 15. This intersection is the union of two elements of the cone. Notice that the elements of the cone are also the paths along which one would measure the distance between point A and B on a cone, if a straight ruler must be placed on the surface of the cone such that the ruler touches all points between the points used to determine the distance. Thus, if A and B are on the same element (ray), then the ruler simply lies along that element (ray). However, by this rule of measurement, to measure from A to B, where A and B are not on the same element (ray), the apex of the cone must be used in this case as the intermediate measuring point, that is, A to O and then O to B.

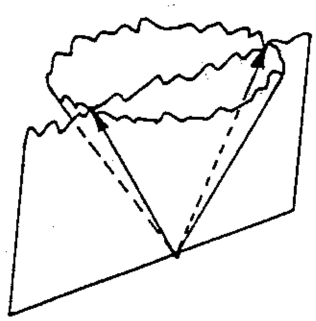


Figure 15.

Using the segments formed from these lines, the $\Delta ABC$ on the cone appears in Figure 16, which when viewed from the apex of the cone, in the direction of the centerline of the cone (that is the projection in the xy-plane), is just the $\Delta ABC$ in the radial plane.

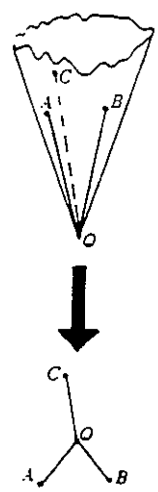
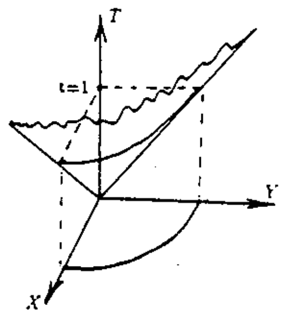



Figure 16.

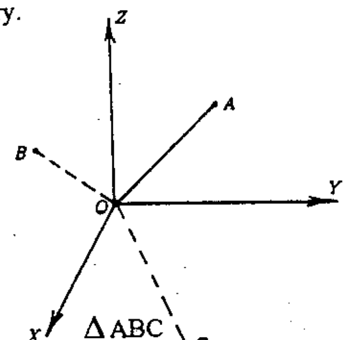
Since these geometries are similarly generated by intersections with flat planes, it is important to mention some of their similarities and differences as axiomatic systems. The axioms which fail in the geometry of the sphere are: the "two points determine a unique line" incidence axiom, the very important Ruler Axiom, and the SAS Axiom. However, the powerful Plane Separation Axiom is true in the sphere. The Radial Plane also fails to satisfy the "two points" incidence axiom and SAS. The interesting difference is that it satisfies the Ruler Axiom and fails to satisfy the Plane Separation Axiom. Hence, the Radial Plane does serve as a legitimate alternative to the model for the sphere.

The exciting observation that can be made, in the following example, is that these elements of the cone are the graphs of the paths of the photons along the three-dimensional Minkowski light cone [2, page 161]. The Minkowski light cone is the graph in space-time coordinates of the position, in space, of a photon produced by a flash of light as a function of time. Suppose that the light travels at one unit distance per one unit time. If the light flashes at the origin in two-space at time, $t = 0$, then at $t = 1$, the photons form a ring in the plane of radius 1, centered at the origin. Since the speed of light is constant, this event is recorded on the Minkowski light cone in three-dimensional space-time coordinates, as shown in Figure 17.




Photon Ring
Figure 17.

With this induced geometry, the projection to the plane of this Minkowski light cone, is the geometry of the radial plane. Extending this idea to the Minkowski light cone in four-dimensional space-time, where the light flashes at $t = 0$ in three-space, we see that the radial geometry in three-space is the projection of the four-dimensional Minkowski light cone geometry, from the apex in the direction of the centerline. Accordingly, Figure 18 illustrates the projection of a $\Delta ABC$ from the four-dimensional light cone, three space coordinates and the time coordinate, into the three-dimensional radial geometry.




$\Delta$ ABC
Figure 18.

Also, depending on your version of reality, events in the past (that is, on the negative time cone) could be represented by multiple representations of points in the radial geometries using + and - subscripts. for example, the $\square A_+B_-C_+D_+$ has vertices A, C, and D on the future cone and the vertex B is on the past cone, as in Figure 19.

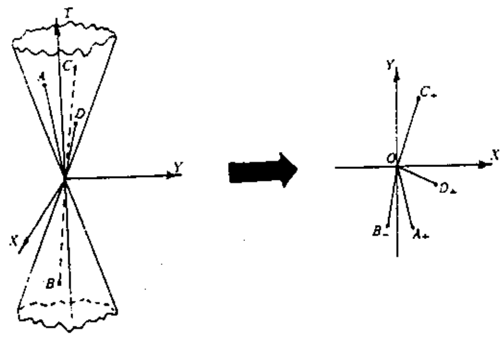



Figure 19.

In closing, I hope that the readers of this little note will enjoy tinkering with this geometry as much as I have enjoyed its development. When you discover some neat fact that I have missed, please share it with me for inclusion in future communications. I do not think this is just another artificial example of a geometry. I think it is as important as the classic example of the geometry of the sphere. It is naturally generated from intersections of planes and cones, in a manner analogous to the sphere. However, it satisfies different axioms than the sphere, and thus provides an alternative view of the interactions of axiomatic systems, as reported in the comparison earlier.

I would like to thank my friends, Professor Jose Garcia, Escuela de Matematica, Universidad de Costa Rica and Professor Stephen Fulling, Department of Mathematics, Texas A&M University for introducing me to the Minkowski light cone.

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# FOCUS ON FACTORING

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Occasionally every teacher encounters a student who seems completely baffled by a particular algebraic skill, such as factoring trinomials. Such a student will often benefit by following a structured method while he or she is struggling to learn the skill. The premise of this article is that even the weakest algebra student can succeed at factoring if presented with a systematic procedure that can be applied to every case. With that in mind, this article presents a collection of classroom tested factoring methods for trinomials in the form $ax^2 + bx + c$ where a, b and c are integers and $a > 0$.

AC Method

The first of these methods for factoring trinomials is called the AC Method. This method is based on the following theorem:

> $ax^2 + bx + c$ is factorable if there exist integers m and n such that $mn = ac$ and $m + n = b$.

Using this theorem, consider the factorization of $24x^2 - 7x - 6$. Integers m and n are needed such that $mn = (24)(-6) = -144$ and so that $m + n = -7$. Begin by factoring -144 and considering the sum of each pair of factors. Since both the sum and the product of the factors must be negative, only the factors of -144 which satisfy both of these conditions have to be considered. (Note that this excludes cases where both factors have the same sign and cases where the factor with the smaller absolute value is negative. These cases would produce positive sums.)

**Table 1**

| Product | Sum |
|---|---|
| (-144) (1) | -143 |
| (-72) (2) | -70 |
| (-48) (3) | -45 |
| (-36) (4) | -32 |
| (-24) (6) | -18 |
| (-18) (8) | -10 |
| (-16) (9) | -7 |

According to the theorem, the trinomial is factorable because there exist integers -16 and 9 whose sum is -7 and whose product is -144. Now rewrite $24x^2 - 7x - 6$ as $24x^2 - 16x + 9x - 6$ (encouraging students to write the negative term first to avoid sign errors), and factor by grouping. (This is a technique which few high school algebra texts introduce prior to trinomial factoring but which is presented early in the study of factoring in developmental mathematics courses at the college level.)

$$24x^2 - 16x + 9x - 6$$

$$= 8x\,(3x - 2) + 3(3x - 2)$$

$$= (3x - 2)\,(8x + 3)$$

Students who were previously successful with factoring by grouping not only tended to have a high success rate with this method, but were later able to adopt the trial and error method as a means of increasing factoring speed.

Box Technique

The second method for factoring trinomials to be considered is called the Box Technique. The Box Technique is based on a multiplication table and the theorem used for the AC method. This method works well with four term polynomials that are traditionally factored by grouping, as well as with trinomials in the for $ax^2 + bx + c$ $(a \neq 0)$ that are traditionally taught by the trial and error method. Consider first the binomial product $(x + 7)\,(x + 5)$.

This product yields $x^2 + 7x + 5x + 35$ by the FOIL method, and simplifies to $x^2 + 12x + 35$. Now multiply the same factors using a multiplication table (see Figures 1 and 2),

| MULTIPLY | **x** | **+5** |
|---|---|---|
| **x** | | |
| **+7** | | |

Figure 1.

and note the properties that occur.

| MULTIPLY | **x** | **+5** |
|---|---|---|
| **x** | $\mathbf{x^2}$ | **5x** |
| **+7** | **7x** | **35** |

Figure 2.

1. The products of the diagonal shaded boxes are equal: $(x^2)\ (35) = (7x)\ (5x)$

2. The sum of the diagonal boxes containing the linear terms produces the linear term of the trinomial: $7x + 5x = 12x$

3. The terms of the original factors (bold faced characters) represent the greatest common factor for that particular row or column.

To see how these properties can be used to factor a trinomial, consider $4x^2 + 16x + 15$. Begin by locating the quadratic term and constant term in the shaded boxes, with the quadradic term being in the center-most box (see Figure 3).

| MULTIPLY | | |
|---|---|---|
| | $4x^2$ | |
| | | 15 |

Figure 3.

Separate 16x into two addends whose product is $60x^2$ (remember, products of diagonals must be equal), and place these two addends in the remaining diagonal boxes. The terms to consider are:

**Table 2**

| Terms | Sum | Product |
|---|---|---|
| 15x, x | 16x | $15x^2$ |
| 14x, 2x | 16x | $28x^2$ |
| 13x, 3x | 16x | $39x^2$ |
| 12x, 4x | 16x | $48x^2$ |
| 11x, 5x | 16x | $55x^2$ |
| 10x, 6x | 16x | $60x^2$ |

Since 10x and 6x produce the desired sum of 16x, as well as the desired product of $60x^2$, the search is terminated. Place these two terms in the remaining pair of diagonal boxes (see Figure 4).

| MULTIPLY | | |
|---|---|---|
| | $4x^2$ | 6x |
| | 10x | 15 |

Figure 4.

Now factor the greatest common factor from each row and column (see Figure 5).

| MULTIPLY | **2x** | **+3** |
|---|---|---|
| **2x** | $4x^2$ | 6x |
| **+5** | 10x | 5 |

Figure 5.

The factorization of $4x^2 + 16x + 15$ is now found in the first row and first column: $(2x + 3)(2x + 5)$.

Now factor $9x^2 - 23x - 12$ by the box technique, and observe the behavior of negative terms in a trinomial. After placing the quadratic term and constant term in the appropriate boxes (see Figure 6), note that this cross product is $=108x^2$.

| MULTIPLY | | |
|---|---|---|
| | $\mathbf{9x^2}$ | |
| | | **-12** |

Figure 6.

The terms whose sum is -12x must also produce a cross product of $-108x^2$. Sums whose terms are both negative or both positive can be excluded since the product must be negative. Also, the term with the largest absolute value must be negative to guarantee a negative sum. Thus, the sums to be considered are:

**Table 3**

| Terms | Sum | Product |
|---|---|---|
| -24x, x | -23x | $-24x^2$ |
| -25x, 2x | -23x | $-50x^2$ |
| -26x, 3x | -23x | $-78x^2$ |
| -27x, 4x | -23x | $-108x^2$ |

The desired sum is -27x + 4x. Before factoring the greatest common factor from each row and column, the box will look like the box in Figure 7.

| MULTIPLY | | |
|---|---|---|
| | $9x^2$ | 4x |
| | -27x | -12 |

Figure 7.

After factoring the GCF from each row and column, the factors of $9x^2$ - 23x - 12 are determined to be (9x + 4) (x - 3). (See Figure 8.)

| MULTIPLY | **9x** | **+4** |
|---|---|---|
| **x** | $15x^2$ | 4x |
| **-3** | -27x | -12 |

Figure 8.

Students who preferred this technique did so because they felt it was easier to find terms that yield a specific sum than to search for factors that yield a specific product. This was especially true in the cases where the trinomials had a large leading coefficient and/or constant term.

Substitution Technique

The last factoring method to be considered is the substitution technique. Since students rarely seem to have difficulty factoring trinomials of the form $x^2 + dx + e$ regardless of the size of d and e, the purpose of this technique is to allow the student to convert every trinomial of the form $ax^2 + bx + c$ (a>0) to the form $x^2 + dx + e$ through substitution. To apply this technique, rewrite the trinomial $6x^2 - 31x + 35$ to produce a perfect square for the leading coefficient. (Multiply by 6 or, in general, by the "a" value). This produces

$6^2x^2 - 6(31x) + 210$. Rewrite as $(6x)^2 - 31(6x) + 210$ and substitute z for 6x. Now factor the trinomial $z^2 - 31z + 210$ by searching for factors of 210 that have a sum of -31. The following algebra steps return the student to the factorization of the original trinomial.

1. Factor the trinomial in z: $(z - 10)(z - 21)$
2. Substitute 6x for z: $(6x - 10)(6x - 21)$
3. Factor GCF from each binomial: $2(3x - 5) \cdot 3(2x - 7)$
   $= 6(3x - 5)(2x - 7)$

Since the original trinomial was multiplied by 6, the factorization of $6x^2 - 31x + 35$ is ((3x - 5)(2x - 7). An alternative arrangement would be to have the student set the problem up as follows:

$$6x^2 - 31x + 35 = \frac{6(6x^2 - 31x + 35)}{6}$$

$$= \frac{(6x)^2 - 31(6x) + 210}{6}$$

$$= \frac{z^2 - 31z + 210}{6}, \textit{when } z \textit{ is substituted for } 6x$$

$$= \frac{(z - 10)(z - 21)}{6}$$

$$= \frac{(6x - 10)(6x - 21)}{6}, \textit{ when } 6x \textit{ is substituted for } z$$

$$= \frac{2(3x - 5) \cdot 3(2x - 7)}{6}$$

$$= (3x - 5)(2x - 7)$$

Conclusions

These techniques have been used successfully with college level developmental mathematics students who have had difficulty with the trial and error method of factoring trinomials. The majority of this group of students seem to prefer the box technique. This may be due to the fact that it is less algebraic in nature than the other methods discussed here. Non-traditional students, who may have seen trial and error factoring in the distant past, seem to opt for one of the other two methods, perhaps because they feel that these methods expedite the factoring process. Many of the students who are adept at trial and error factoring will be reluctant to leave behind comfortably familiar territory in order to try a new technique. However, even students who are comfortable with trial and error factoring are encouraged to try at least one of the structured techniques in order to gain a different viewpoint on the process of factoring trinomials. Thus, each student in the class, regardless of background, will benefit in some way from including these structured formats in a discussion of trinomial factoring.

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