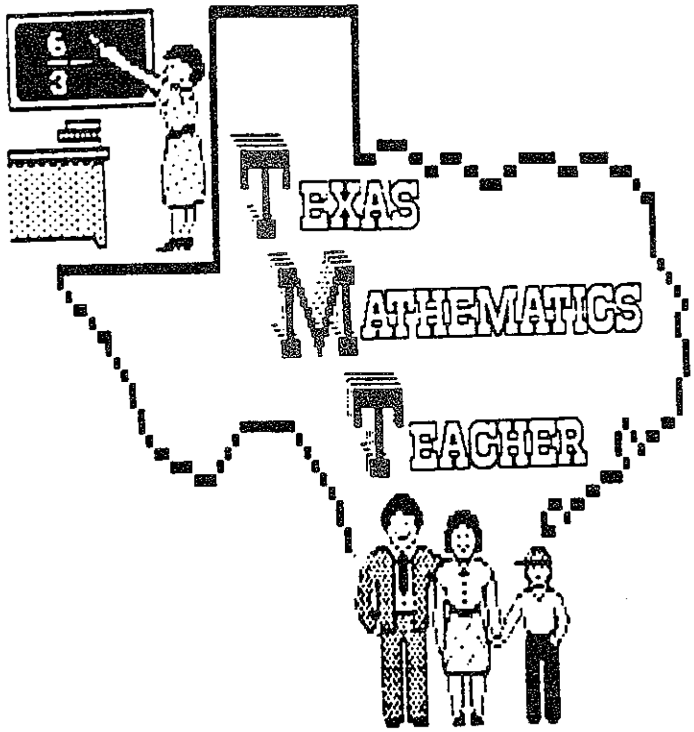

Polynomial Explorations with a Computer Algebra System

Investigating the Sphere

Mathematics Reform for Teacher Education

Emphasizing Number Sense Through
Cooperative Learning Activities

MARCH 1991

EDITORS

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TEXAS MATHEMATICS TEACHER

TEXAS COUNCIL OF TEACHERS OF MATHEMATICS

Affiliated with the National Council of Teachers of Mathematics

March 1991

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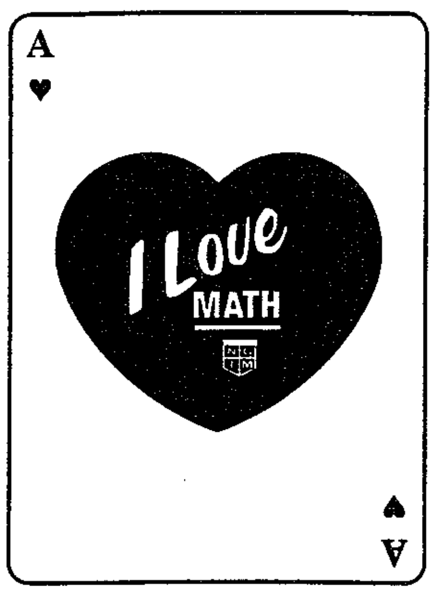
President's Message

"Math is useless to an artist" the voice whined on the other end of the phone line. It was 9:00 at night and I had just returned home from judging a science fair. Since the day had started at 7:00 with a make-up exam for a student who had been ill, I was probably more short tempered than I would have been if I had received this phone call at a more reasonable hour. The mother who spoke on the other end of the line was trying to justify why her daughter felt it necessary to drop senior mathematics during the last semester of her last year and convince me not to write the admissions office of the colleges where her daughter was applying to inform them that this was no longer the same applicant.

Is math useless to an artist? I think not, but I did not convince either the mother or the daughter. As I sit and write this, I am angry with the mother, because she is so quick to defend the actions of a second semester senior looking for an easier course for spring semester and because I always have this conversation about daughters and never about sons. Would I have had the conversation if the student were the mother's son? I think not. We sell our daughters short because we do not hold them to the same academic rigor as we do our sons. We, as a society, are ready to excuse women from a quantitative course of study. A recent study points out that Asian mothers expect their daughters to study hard and achieve in mathematics, and they do. By contrast, American mothers do not expect their daughters to achieve in mathematics, and they don't. The nineties are better than the fifties. Women who came of age in the fifties were so excluded from the mathematics and science curriculum of the early sixties, that a few years ago the National Science Foundation sponsored graduate fellowships for women who earned Bachelor of Science degrees in mathematics and science in the early sixties to return to school and earn a graduate degree. Today, schools invite, cajole and bribe young women to remain in mathematics courses during their entire pre-college education only to be thwarted by mothers who allow their daughters to opt for short term gratification.

Karen S. Hall



POLYNOMIAL EXPLORATIONS WITH A COMPUTER ALGEBRA SYSTEM

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Computer algebra systems (CAS) are finding their way into the mathematics curriculum at every level. It is thought that through appropriate use of CAS we can revitalize undergraduate mathematics, increase conceptual understanding, inspire our students and assist "weak students." The (CAS) software Maple is available for Macintosh computers and will run on systems with as little as one megabyte of RAM memory. Existing Macintosh-Plus computer laboratories can be networked to run Maple on a file server. Also, a new student version of Maple will soon be released. This article illustrates some ideas involving polynomials: expansion, factorization, root finding and division which were used by the author in our course titled "Mathematical Computation for Teachers" here at California State University Fullerton.

Factors and roots of polynomials.

Factoring polynomials by hand requires several attempts using trial coefficients. It is time consuming for students and limits the nature of textbook problems to "cooked-up" examples. With CAS, students can work with more complicated examples. Maple has a built-in procedure *factor* which can be used for polynomials with integer or rational coefficients. Enter the polynomials *p2, p3,* and *p4,* into a Maple session:

p2 :=x^2 + x - 6;

*p3 :=x^3 - 2*x^2 - x + 2;*

*p4 :=x^4 - 6*x^3 + 3*x^2 + 26*x - 24;*

The response by the computer is to echo what was typed in:

$$p2 := x^2 + x - 6$$

$$p3 := x^3 - 2\,x^2 - x + 2$$

$$p4 := x^4 - 6 \cdot x^3 + 3\,x^2 + 26\,x - 24$$

Now we envoke Maple's *factor* procedure and type the commands *p2:=factor(p2); p3:=factor(p3); and p4:=factor(p4)*. The results displayed by Maple are:

$$p2 := (x + 3)\,(x - 2)$$

$$p3 := (x + 1)\,(x - 1)\,(x - 2)$$

$$p4 := (x - 4)\,(x + 2)\,(x - 1)\,(-3 + x)$$

The factorization of a polynomial p(x) can be used to determine the roots of the equation p(x) = 0, or they can be obtained directly by invoking the built-in *solve* procedure. In our case, all we need to do is issue the commands *set2:=solve(p2=0); set3:=solve(p3=0,x);* and *set4:=solve(p4=0,x);*

set2 := -3, 2

set3 := -1, 1, 2

set4 := 4, -2, 1, 3

The solutions which were stored in *set2*, *set3*, and *set4* are easily seen to produce a zero in an appropriate factor of the polynomial *p2*,*p3* and *p4*, respectively.

High resolution graphics are included in Maple. Graphs for the polynomials are created by typing *plot(p2,x=-4..4,y=-7..13); plot(p3,x=-2..3,y=-10..10); and plot(p4,x=-3..5,y=-40..60);* They are shown in Figures 1,2, and 3, respectively.

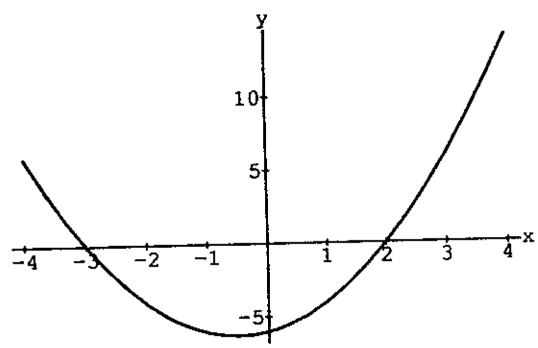

Figure 1. The graph of $y=p_2(x)=x^2+x-6$.

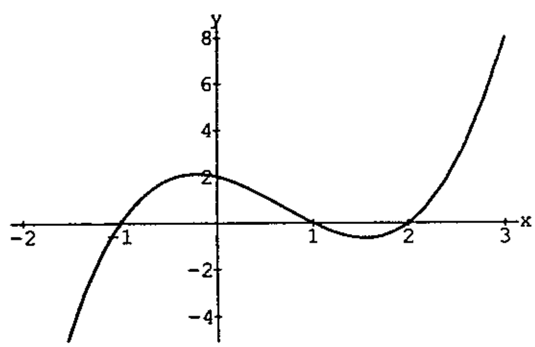

Figure 2. The graph of $y=p_3(x)=x^3-2x^2-x+2$.

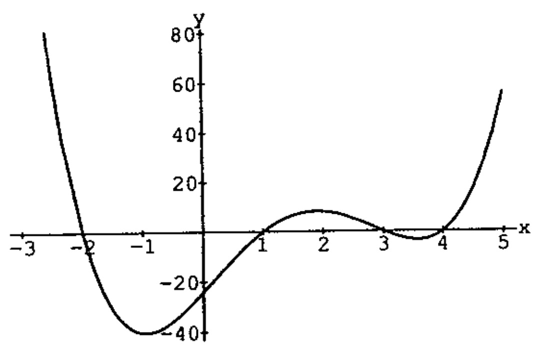

Figure 3.
The graph of $y=p_4(x)=x^4-6x^3+3x^2+26x-24$.

Numberical solutions.

Accurate numerical approximations for roots of equations are often needed, especially when the roots are not integers or rational numbers. For these cases, the Maple procedure *fsolve* can be used. Looking at the graph of $y=p_3(x)$ in Figure 2, we can determine that the lines $y=1$ and $y=2$ will intersect the curve in three points. But the line $y=3$ will lie above the "hump" near the y-axis and thus the line $y=3$ will intersect the curve in only one point. The locations of these points of intersections can be obtained by solving the equations $p_3(x)=1$, $p_3(x)=2$ and $p_3(x)=3$, respectively. These solutions can be found numerically by issuing the commands *fsolve(p3=1,x);fsolve(p3=2,x);* and *fsolve(p3-3,x);* The computer will find and print out the solution sets for the equations $p_3(x)=1$, $p_3(x)=2$ and $p_3(x)=3$:

-.8019377360, .5549581319, 2.246979604

-.4142135623, 0, 2.414213563

2.546818277

In a similar fashion, we can look at the graph of $y=p_4(x)$ in Figure 3, and see that the line y = c (where $c>0$) will intersect the curve in four points when c is small and intersect the curve in only two points when c is large. The cross-over will occur at the local maximum near x=2. Investigation into these matters is a good laboratory exercise for students. With CAS students have time to make these explorations. Issue the commands: *fsolve(p4-7,x);fsolve(p4=8,x);*and *fsolve(p4=9,x);* and the response by the computer is the solution sets for $p_4(x)=7$, $p_4(x)=8$ and $p_4(x)=9$:

-2.073892604, 1.559301596, 2.245465479, 4.269125529.

-2.083872359, 1.786801815, 2.000000000, 4.297070545

-2.093722365, 4.323548705

These investigations have illustrated how the number of solutions to a polynomial equation p(x)=c changes when the parameter c is varied.

Explorations in factorization.

Students can perform meaningful experiments with a CAS tool. Exercises in almost every algebra course involve factorizations of the form $p(x) = x^n - a^n$ for n=2 and 3, and higher powers of n are left for a challenge. But usually this is a time consuming task involving pages of pencil and paper computations. Maple is a powerful assistant which enables students to experiment into the nature of these mathematical solutions. Polynomials up to degree n-8 are entered into the session: *p2:=x^2-a^2;p3:=x^3-a^3;p4:=x^4-a^4;p5:=x^5-a^5;p6:=x^6a^6;* *p7:=x^7-a^7;p8:=x^8-a^8;* Then the commands to factor the polynomials are given: *p2:=factor(p2);p3:=factor(p3);p4:=factor(p4);* *p5:=factor(p5); p6:=factor(p6); p7:=factor(p7); p8:=factor(p8);* The computer's response is:

$p2 := (x - a)\ (x + a)$

$P3 := (x - a)\ (x^2 + x\,a + a^2)$

$p4 := (x - a)\ (x + a)\ (x^2 + a^2)$

$p5 := (x - a)\ (x^4 + x^3 a + x^2 a^2 + x\,a^3 + a^4)$

$p6 := (x - a)\ (x + a)\ (x^2 + x\,a + a^2)\ (x^2 - x\,a + a^2)$

$p7 := (x - a)\ (x^6 + x^5 a + x^4 a^2 + x^3 a^3 + x^2 a^4 + x\,a^5 + a^6)$

$p8 := (x - a)\ (x + a)\ (x^2 + a^2)\ (x^4 + a^4)$

Students can spend quality time reflecting on their findings and make conjectures regarding the patterns they discover. One conjecture that can be made is that there are two factors if and only if the power n is a prime number, i.e. in the cases n = 2, 3, 5, 7. Such conjectures can be proven, but require skill in the branch of mathematics known as "number theory."

Binomial expansions.

Binomial expansions have always been fascinating exercises for students. The theory leads them into the study of binomial coefficients and Pascal's triangle. Maple can help students create expansions of any order. The first few are obtained by typing *expand((x+a)^2); expand((x+a)^3); expand((x+a)^4); expand((x+a)^5); expand((x+a)^6); expand((x+a)^7);*

$x^2 + 2\,x\,a + a^2$

$x^3 + 3\,x^2 a + 3\,x\,a^2 + a^3$

$x^4 + 4\,x^3 a + 6\,x^2 a^2 + 4\,x\,a^3 + a^4$

$x^5 + 5\,x^4 a + 10\,x^3 a^2 + 10\,x^2 a^3 + 5\,x\,a^4 + a^5$

$x^6 + 6\,x^5 a + 15\,x^4 a^2 + 20\,x^3 a^3 + 15\,x^2 a^4 + 6\,x\,a^5 + a^6$

$$x^7 + 7x^6a + 21x^5a^2 + 35x^4a^3 + 35x^3a^4 + 21x^2a^5 + 7xa^6 + a^7$$

General theory of polynomials.

The binomial expansions are a prelude to the theory of polynomial forms that can be explored. For example, suppose that the roots of a cubic polynomial p(x) are r_1=-2, r_2=1, r_3=3 then it is easy to construct p(x) = $c_3x^3 + c_2x^2 + c_1x + c_0$ with the commands:

p := (x + 2)(x - 1)*(x - 3);*

p := (x + 2) (x - 1) (x - 3)

p := expand(p);

$$p := x^3 - 2x^2 - 5x + 6$$

In this example, $c_0=(-r_1)(-r_2)(-r_3)=(2)(-1)(-3)=6$ is the product of the negatives of the roots. The coefficient of x is $c_1 = -5 = (1)(3) + (-2)(1)$ and the coefficient of x^2 is the negative of the sum of the roots, i.e. $c_2= -(r_1 + r_2 + r_3) = -(-2+1+3)=-2$. A general result can be investigated by starting with the roots $\{_{rk}\}$ and expanding the polynomials $p_2(x) = (xr_1)(x-r_2)$, $p_3(x)=(x-4_1)(x-r_2)(x-r_3)$, and $p_4(x)=(x-r_1)(x-r_2)(x-r_3)(x-r_4)$. The coefficient c_0 is always the product of the negatives of the roots and c_{n-1} is the negative of the sum of the roots. Maple can be used to help students explore these ideas.

p2 := collect(expand((x - r1)(x-r2)), x);*

$$p2 := x^2 + (-r2 - r1)\,x + r1\,r2$$

p3 := collect(expand((x - r1)(x - r2)*(x - r3)), x);*

$$p3 := x^3 + (-r3 - r2 - r1)\,x^2 + (r2\,r3 + r1\,r3 + r1\,r2)\,x - r1\,r2\,r3$$

p4 := collect(expand((x - r1)(x - r2)*(x - r3)*(x - r4)), x);*

$$p4 := x^4 + (-r4 - r3 - r2 - r1)\ x^3$$

$$+ (r3\ r4 + r2\ r4 + r2\ r3 + r1\ r4 + r1\ r3 + r1\ r2)\ x^2$$

$$+ (-r2\ r3\ r4 = r1\ r3\ r4 - r1\ r2\ r4 - r1\ r2\ r3)\ x$$

$$+ r1\ r2\ r3\ r4$$

Polynomial quotients.

Any discussion about polynomials will usually end with the topic of quotients. Long division techniques may serve as good "busy work" but ideas are soon lost in a maze of calculations. Maple can easily handle the calculations. For an easy example let us divide $p(x) = x^3 + 5x^2 - 7x + 14$ by $d(x) = x - 3$.

*p := x^3 + 5*x^2 - 7*x + 14;*

$$p := x^3 + 5\ x^2 - 7\ x + 14$$

d := x - 3;

$$d := -3 + x$$

The commands for finding the quotient and remainder are *quo* and *rem* respectively.

q := quo(p,d,x);

$$q := x^2 + 8\ x + 17$$

r := rem(p,d,x);

$$r := 65$$

We can check our work by forming $qd + r$ and comparing it with p.

```
m := q* + r;
```

$$m := (x^2 + 8\ x + 17)\ (-3 + x) + 65$$

```
expand(m);
```

$$x^3 + 5\ x^2 - 7\ x + 14$$

To illustrate the power of Maple we can of course perform a complicated computation:

```
p := x^5 - 7*x^4 + 2*x^2 + 7*x - 9;
```

$$p := x^5 - 7\ x^4 + 2\ x^3 - 4\ x^2 + 7\ x - 9$$

```
d := x^2 + 3*x - 4;
```

$$d := x^2 + 3\ x - 4$$

```
q := quo(p,d,x);
```

$$q := x^3 - 10\ x^2 + 36\ x - 152$$

```
r := rem(p,d,x);
```

$$r := 607\ x - 617$$

```
m := d*q + r;
```

$$m := (x^2 + 3\ x - 4)\ (x^3 - 10\ x^2 + 36\ x - 152) + 607 \cdot x - 617$$

```
expand(m);
```

$$x^5 - 7\ x^4 + 2\ x^3 - 4\ x^2 + 7\ x - 9$$

Conclusion.

We have shown how polynomials can be manipulated symbolically on a personal computer. Using CAS makes mathematics interesting and enjoyable for both students and teachers. Many other topics in elementary mathematics can be investigated in depth with CAS. The reader is encouraged to try a computer algebra system and discover its capabilities. Other popular systems are DERIVE, Mathematica and muMATH.

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INVESTIGATING THE SPHERE

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Instruction in problem solving often involves seeking alternative procedures. Teaching units on the standard two- and three-dimensional geometric shapes offer an interesting opportunity for students to search for alternative models that demonstrate their various formulas. Such activities result in greater intuitive understanding of these formulas for the students involved.

As an example, the sphere with its surface area and volume formulas provides challenging problems for the secondary student. Investigations with the sphere's formulas can take several different directions, depending on the imaginations of the students involved. Discussion of various discoveries made by the students results in a keener understanding of the relationships the formulas represent. One such investigation is presented below.

> <u>Problem One</u>: "Demonstrate by means of a concrete model how the surface area of a sphere is a function of the sphere's radius r, as expressed by the formula, S.A. = $4\pi r^2$."
> [Each team of 3-4 students is given a small, hollow, soft, plastic or rubber ball (diameter approx. 5 inches) to help them visualize the problem and to use as they wish during their investigation. Most hollow, toy balls work perfectly, especially if they maintain their curved form to a small degree after begin cut in half.]

When a team of students first encounters this problem, the students are immediately faced with the dilemma of trying to relate an interior <u>linear</u> measure (r) to a surface area or <u>square</u> measure. Many teacher hints are needed to guide them toward devising a strategy or plan to follow. The most effective suggestions call attention to the possibility of cutting the ball apart and to the surface area formula as a product of several factors, including the radius r. Once the students begin to focus on the formula factors, their investigation progresses. The most common models constructed are presented below.

Solution 1: Viewing the surface area formula as "4times πr^2". The surface area is considered as equivalent to the area of four great circles of the sphere. [Note: A 'great circle' of a sphere is any circle, formed by a plane passing through the sphere, which has the same radius as the sphere and therefore a maximum circumference.] The radius measure of the sphere is obtained by measuring the circumference of a visually located, great circle of the ball (sphere) or the diameter of a hemisphere (half of the ball) after the soft, hollow ball has been cut in half.

At this point two directions are possible:

1.) By this time the students realize that it is sufficient to work with only half of the ball and its surface area, $2\pi r^2$. Two great circles of radius r are drawn on paper or fabric, usually by tracing around the cutting edge of the ball's hemisphere. The two great circles are cut in half, then taped to the hemisphere's surface. Some of the halves must be cut into smaller pieces in order to cover gaps between the larger pieces. Of course, no matter how the circles are cut, there will still be some gaps and buckling in the surface covering, and students should discuss this as a limitation of their model. However, the students also recognize intuitively that, if infinitely small pieces could be cut (a little limit-taking, if you please), the surface would indeed be covered smoothly by the two great circles. Conclusion: Half the surface area of a sphere equals the area of two great circles of the sphere ($\frac{1}{2}$ S.A. = $2\pi r^2$). [See Figure 1.]

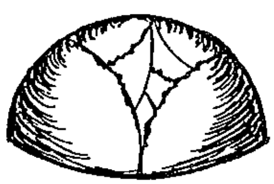

Figure 1. Hemisphere "Covered" with Flat Sections from Great Circles

2.) One great circle of radius r is drawn on paper. A hemisphere of the ball is cut in half to form a fourth of the sphere. This fourth of the ball's surface is then cut into small pieces and taped onto the paper circle so that most of the interior of the circle is covered. Again, there will be gaps and bucklings because of the inflexibility of the soft plastic or rubber material used, but the students view this as the result of inaccuracies in measuring, not as a contradiction of the original 'sphere-great circle' relationship. Conclusion: The surface area of a fourth of a sphere equals the area of one great circle of the sphere (¼ S.A. = πr^2). [See Figure 2.]

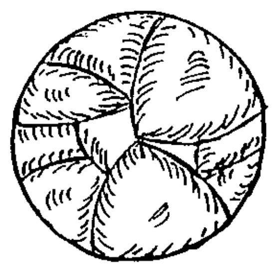

Figure 2. Great Circle "Covered" with Curved Sections from Hemisphere

Solution 2: When the formula is seen as "4π times r^2", the r^2 is interpreted as the area of a square with a side measure equal to the radius r of the sphere. Once the radius measure is found by methods described previously, several squares of side r are drawn and cut out of paper. Students have great difficulty viewing 4π as a quantity at first, but soon realize that if 4π of the r-squares are supposed to cover the entire ball, then 2π or about 6.28 of the squares should cover half of the ball's surface. (The presence of pi in the formula provides students with needed experience in estimation.) They tape the squares onto the outer surface of a hemisphere of the ball, cutting some of them to fill in gaps where necessary. Six squares plus part of another square are indeed needed to cover the surface with only a small number of gaps or bucklings remaining. Conclusion: Half of the surface area of a sphere is equal to 2π squares having areas of r^2 (½ S.A. = $2\pi[r^2]$). [See Figure 3.]

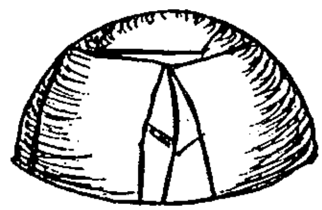

Figure 3. Hemisphere "Covered" with Squares

<u>Solution 3</u>: Viewing the formula as "4 times πr times r," the area of a rectangle can be used. A rectangle of length πr and width r is drawn on paper, using the ball's radius r. Half of a hemisphere of the ball is cut into small pieces and taped onto the rectangle's interior to approximate a covering. Conclusion: A fourth of the surface area of a sphere is equal to the area of a rectangle of length πr and width r ($\frac{1}{4}$ S.A. = $[\pi r][r]$). [See Figure 4.]

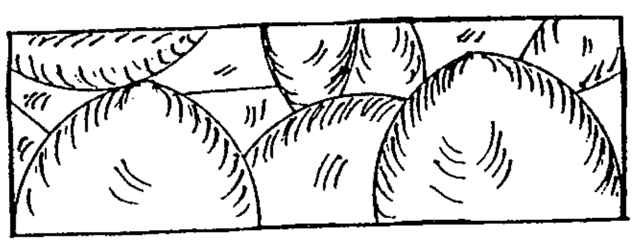

Figure 4. Rectangle "Covered" With Sphere Sections

Each of the above investigations requires about two hours of team effort, depending on the amount of previous problem-solving experience students have had. A half-hour per day for 3-4 days works well, thus allowing class time each day for other topics to be covered as well. Once the teams, however, have constructed their concrete models, they should present their "products" to the class. One person from each team should give a 3-5 minute presentation to the class, showing the team's model and explaining the procedure used. The effects of any approximations made, as reflected in the gaps and buckling of the surface covering, should also be mentioned. All team models should be analyzed and compared. In particular, the original expectations of each team as the assignment was first attacked and the clarity with which each team's method relates the sphere's radius to its surface area, should be freely discussed by the entire class.

Another investigation of the sphere involves the volume formula. This problem, too, can take several different directions, depending on the grouping of the formula's factors. Only one method or solution will be discussed in detail at this time.

> Problem 2: "Demonstrate by means of a concrete model how a sphere's volume is a function of its radius, as expressed by the formula, $V = {}^4/_3\pi r^3$."
> [Students are again furnished a small, hollow ball to help them visualize the relationships involved.]

Solution: Consider the volume formula as "4 times $\frac{1}{3}\pi r^3$" or "4 times $\frac{1}{3}[\pi r^2][r]$". This implies that the sphere's volume is four times the volume of a cone with base πr^2 and height r. Using a hemisphere of the team ball, a great circle is drawn on paper and the sphere's radius r computed from the great circle's circumference. This drawn circle will serve as the pattern for the base of the desired cone. Another circle of radius $r\sqrt{2}$ (r = computed radius of the sphere) is drawn on heavy paper and cut out. The value of $r\sqrt{2}$ is the slant height needed to produce a cone of height r. [See Figure 5.] This second circle is shaped into a cone and adjusted until it fits onto the great circle pattern of radius r as its base. The cone's overlapping edges are then taped together. The finished cone can now be used to fill a hemisphere of the hollow ball with water. Because a single hemisphere is being used, only two cones of water will be needed. Conclusion: The volume of a hemisphere of radius r equals twice the volume of a cone of base πr^2 and height r (½ Volume of sphere $= 2 \times \frac{1}{3}[\pi r^2][r]$). [See Figure 6.]

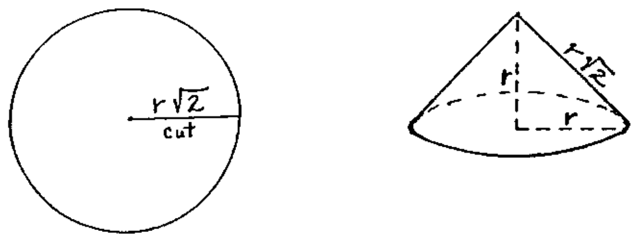

Figure 5. Pattern for cone.

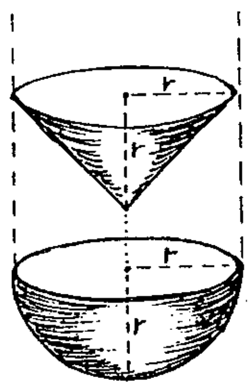

Figure 6. Hemisphere and corresponding cone.

Another solution to the volume problem might involve counting the number of cubed boxes of edge r (consider V and $\frac{4}{3}\pi[r^3]$) needed to fill each hemisphere with sand. Cardboard cylinders of base πr^2 and height r (consider V as $4/3[\pi r^2][r]$) might also be used to fill the hemisphere with sand or water. Inversely, the hemisphere might be packed with clay, then this clay remolded into cylinders of the proper size. There should be enough clay taken from the hemisphere to form one and a third such cylinders.

As with the surface area problem, the different volume models and methods should be discussed with the entire class. Students should realize that estimation has been applied throughout the problem, so exactness should not be expected. Encountering the volume formula in such a problem solving setting will greatly strengthen the students' decision-making skills and their understanding of the prerequisite geometric concepts.

Such investigations as the surface area and volume problems presented in this article can be derived from most of the area and volume formulas in geometry. The analysis and regrouping of the factors used in these formulas help students become more aware of the various planar and spatial relationships represented by the formulas. Teachers should encourage their students to view formulas in different ways in order to increase their perception and understanding of the roles of formulas in everyday applications.

MATHEMATICS REFORM FOR TEACHER EDUCATION

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Since the National Council of Teachers of Mathematics published the Curriculum and Evaluation Standards for School Mathematics in April 1989, (NCTM, 1989) mathematics educators have devoted much time and effort to disseminating the spirit and contents of that document. Mathematics teachers throughout the country are becoming aware of the objectives and topics which are essential ingredients to updating and improving current mathematics teaching practices in this country.

Certainly, American students are not achieving the success in mathematics expected by the educational system or the public. Pupils in the United States fare poorly when compared to other nation's scores on math tests (McKnight et al., 1987). Current mathematical achievement of United States students is nowhere near the level required to sustain our nation's leadership in the global, technological society of the late twentieth and early twenty-first centures (National Research Council, 1989). The need for a document such as the Standards as well as reports on the status of mathematics teaching today such as Everybody Counts (National Research Council, 1989) are timely and critical to the nation's future... "for lack of mathematical power, many of today's students are not prepared for tomorrow's jobs" (National Research Council, 1989). American students need the type of education called for by the National Council of Teachers of Mathematics and the National Research Council in which students learn to problem solve, reason, and develop confidence in their mathematical ability. Hopefully, as these objectives are accomplished, American students will succeed in mathematics as individuals and as members of an increasingly complex and competitive world.

Not only must kindergarten through twelfth-grade students be educated with high-quality pedagogical techniques and the range of content topics needed to reach the aforementioned goals, future mathematics teachers must also be thoroughly educated in the content and the intent of mathematics teaching reforms. This latter group of individuals is not as often addressed as a population to be educated in the call for reform as is the pre-college students. Pre-service teachers, that is, undergraduate education students, are those who are expected to examine and analyze the most recent reports of research, methodology and trends in mathematics education. These students are in the position of learning not only the theoretical foundations of mathematics teaching methods, but also have a fresh opportunity to apply those viable ideas, such as described in the Standards, in tomorrow's classrooms.

Future teachers have a unique opportunity to make a change from mathematics teaching techniques of rote and abstract presentations found in some classrooms today. Education students are most recently formally presented with today's ideas which their pupils will need tomorrow. These education students will be teaching pupils who will work in the next century and so the former will have the opportunity to truly make a timely difference in this country's educational and economic advancement. As such, it is imperative that the college population be educated in theory and methodology that best fits the needs of their future pupils.

In terms of specifically educating the college students to make the crucial changes necessary in mathematics teaching, research indicates that they need to learn pedagogical strategies using manipulatives just as pupils do when learning mathematics content at the pre-college level (Sherman, 1989). When future teachers learn problem-solving teaching strategies by actually solving problems and become adept at teaching communication skills by communicating, then they will much more likely become proficient in the substance and style of what mathematics teaching should be.

Those who are presently teaching mathematics often claim that they teach as they were taught. Studies indicate that teachers will repeat those methods with which they are familiar and comfortable (Carnegie Foundation, 1986). Since much of the current pool of instructors were educated with pedagogical techniques emphasizing rote and abstract approaches, the style is well established in our society (National Commission on Excellence in Education, 1983). Future teachers will more likely teach math in a problem solving atmosphere based upon discovery of patterns and communication if they have been exposed to that type of instruction during their college education. There would be less likelihood to teach in that manner if they have never experienced modeling and logical thinking in their studies.

Mathematics educators can not rewrite the pre-college mathematics experience of their undergraduate students. However, college mathematics instructors are in the position to provide current, well designed methods and teaching styles for their students at this point in their career. The anxiety-producing and mechanical teaching styles of today's K - 12 classrooms might be eliminated in coming years if pre-service teachers are educated in the manner called for in mathematics reform documents. Then much progress will be made toward breaking the cycle of rote and repetitive instruction presented by teachers who experienced that type of education when they attended school. University level instructors of mathematics education and content courses should provide students with the wide variety of experiences recommended in documents such as the Standards. Opportunities for growth in communication, problem solving, reasoning and a strong foundational understanding of mathematics would then exist at the preservice teacher level of formal education. College mathematics education courses are a vital ingredient in the effort to reform and lift mathematics achievement in this country to levels of excellence to which it can and must strive.

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EMPHASIZING NUMBER SENSE THROUGH COOPERATIVE LEARNING ACTIVITIES

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According to the Commission on Standards for School Mathematics of the NCTM, students who have good number sense (1) have well-understood number meanings, (2) have developed number relationships, (3) have an understanding of the relative magnitudes of numbers, (4) have a knowledge of the relative effect of number operations, and (5) have developed referents for measures of common objects and situations in the environment. If students are to succeed in math, they need to possess good number sense, or "common sense math." Math teachers must utilize activities that incorporate concepts of number sense.

With such a great emphasis being placed on number sense, classroom teachers must provide appropriate activities that address this topic. However, number sense should be developed not as a topic itself, but interwoven throughout the curriculum. There are many possibilities to supplementing a lesson with activities that incorporate the concept of number sense.

An effective means of developing number sense in our math students is through cooperative learning activities. Such activities can be successfully adapted to any age level. The key to effective cooperative learning is the word "cooperative" rather than "competitive" as customarily used in math classrooms (Rosenbaum 1989).

Cooperative learning involves students working in small groups with a particular goal or objective. These cooperative groups are usually heterogeneous and student oriented. The emphasis is on the strategy being used rather than the answer obtained. It is important that teachers encourage every student in the group to contribute to the goal. Answers determined by the group must be agreed upon by the group. Regardless of the nature of the activity, all members should contribute to the welfare of all other members, and success of each member is dependent on success of the group (Behounek, 1988).

Below is a number of sample activities stressing several facets of number sense and at the same time providing a vehicle for cooperative learning experience.

M & M MATH

This activity can be used in the primary grades and focuses on counting, classifying, and working with fractions. Each pair of students is given a cup of M&M's containing 6 brown ones, 5 orange ones, 4 yellow ones, 3 red ones, and 2 green ones. (More or less may be used depending on grade level.) As a team, the students will cooperatively try to complete the worksheet.

SAMPLE WORKSHEET:

1. How many M&M's do you have altogether? ______
2. How many green ones do you have? ______
3. The number of brown ones is how many times the number of red ones? ______
4. How many yellow ones do you have? ______
5. What fraction indicates the relationship of yellow to green? ______ What does this mean? ______________________________
6. Create 2 word problems using the M&M's. The problem may deal with fractions or any basic operation with fractions or any basic operation of arithmetic.

____

____

____

This activity incorporates the components of number meanings and number relationships. Counting manipulatives can help students to visualize the meaning of numbers. Students can discover relationships between numbers by differentiating M&M's by color and using the fraction concept.

PEANUTS ARE FUN

This activity is designed for the middle school grades. Students work in pairs and are given a container of peanuts (10). The teacher will ask students to write their findings on a sheet of paper. The following directions will be given orally by the teacher.

1. Students spread the peanuts on the table in front of them. Working in pairs they decide how they want to divide the peanuts into equal sets. (The teacher may call on some of the students to share their results before going on.)
2. Students then combine the sets. They then subtract 5 from the total and consider the following question. Can the peanuts be put into the same subsets or groups as before? Why/why not? If not, how can they be grouped?
3. The pairs of students then jointly come up with a list of multiplication and/or division facts that they can represent with the peanuts. They can also share with the class.
4. Students now put the peanuts back into the container. Each student selects one peanut from the container and must describe its characteristics (on paper, as specific as possible). Ex. fat, slim, pear-shaped, about one inch long, light-colored, etc. The characteristics on paper will be used to identify the peanut. Peanuts are then placed back into the container and subsequently spread on the table. One student reads his characteristics and the other student tries to determine which peanut is being described. The other student will do the same. Two or three pairs of students will be combined to share characteristics used in determining or finding the peanut.

The peanut activity focuses on the first two components of number sense, the meaning of 30 and relating it to its factors. It also focuses on the fourth component by looking at the effect of subtracting 5 from 30; the fifth component is included by comparing one peanut to a group of peanuts.

STRING METRICS

This activity is easily adapted to primary and/or middle school grades. Its focus is on giving students experience in measuring with a meter-long string. It is most effectively conducted with students working in pairs.

Each pair is provided a meter-long string and, each student is given a measurement worksheet to record his/her estimates and actual measurements.

SAMPLE WORKSHEET:

1. Select various objects in the classroom and estimate the measurement of each in relation to a meter.

Object being measured: ___ ___ ___

My estimation: ___ ___ ___

Partner's estimation: ___ ___ ___

Our measurement: ___ ___ ___

For each object measured, determine whose estimation was the closest to the actual measurement.

2. Name any object in the classroom that is approximately one meter long, one that is half as long as a meter, and one that is twice as long as a meter. Remember these are approximations. Each pair of students will share their findings with the rest of the class.

3. The following measurements are about the students themselves. After measuring, with the help of your partner, write "longer" or "shorter" in the blank provided.

My jump is ___ than a meter.

My step is ___ than a meter.

My arm is ___ than a meter.

Three of my jumps are ___ than a meter.

Ten of my steps are ___ than my jump.

This metric activity incorporates number meanings and number relationships. Measuring various objects and distances allows the students to visualize the meaning of numbers and the relationships between measurements. The fifth component of number sense is also incorporated by students utilizing a referent in measuring objects in the environment.

WHAT'S IN A NUMBER

This activity is designed for the primary or middle school grades. The focus is having students describe numbers in various ways so as to enhance their understanding of numbers.

Each pair of students selects an index card from the "number bag." They write the number on a sheet of paper and must describe their number in 4 ways. (Each partner contributes two.) Then each pair will share their number and its description with the class.

EXAMPLE: 18

3 times the number 6
1½ dozens of eggs
legal age of voting
7 pennies less than a quarter

A variation on this activity would be to have the students make up a word problem with the answer being the number they selected.

Many of the components of number sense are found in the above activity. Students are verbally expressing numbers in different ways which can improve their understanding of the meanings of numbers. Relationships between numbers can be discussed using the students' various expressions. The concept of number magnitude can also be incorporated with some of the expressions, such as "3 times the number 6." An expression such as "7 pennies less than a quarter" could be used to incorporate operations on numbers.

The activities suggested vary in math content, math skill, as well as grade level. All of them do, however, provide an opportunity for students to "experience number sense" in a cooperative learning setting and within the regular math curriculum. By providing a variety of activities such as these, the teacher is helping to insure the student's understanding of math in today's classroom.

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National Council of Teachers of Mathematics

Proclamation

Whereas, mathematical literacy is essential for citizens to function effectively in society; and,

Whereas, mathematics is used every day—both in the home and in the workplace; and,

Whereas, the language and processes of mathematics are basic to all other disciplines; and,

Whereas, our expanding technologically based society demands increased awareness and competence in mathematics; and,

Whereas, school curricula in mathematics provide the foundation for meeting the above needs;

Now, therefore, I, Iris M. Carl, President of the National Council of Teachers of Mathematics, do hereby proclaim the month of April 1991 as

Mathematics Education Month

To be observed in schools and communities in recognizing the increased importance of mathematics in our lives.

In witness thereof, I have hereunto set my hand and caused the corporate seal of the National Council of Teachers of Mathematics to be affixed on this 1st day of September 1990.

President

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