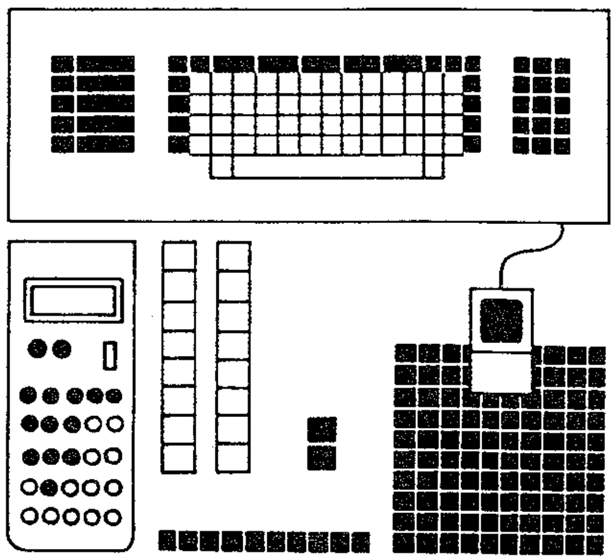
TEXAS MATHEMATICS TEACHER



Introducing the Point of Division Formulas For A Line Segment

Using Computer Algebra Systems For Indeterminate Forms

Informal Assessment of Computational Errors

Triangles With Rational Sines

MAY 1990

TEXAS MATHEMATICS TEACHER, the official journal of the Texas Council of Teachers of Mathematics, is published four times each year, in October, January, March, and May. All manuscripts and editorial correspondence should be addressed to George H. Willson, listed above.

Manuscripts should be neatly typewritten and double spaced, with wide margins, on 8 1/2 " by 11" paper. Authors should submit the signed original and four copies. Illustrations should be carefully prepared in black ink on separate sheets, the original without lettering and one copy with lettering added. As soon as possible after refereeing, authors will be notified of a publication decision. Originals of articles not accepted for publication will be returned to the authors. Two copies of the issue in which an author's manuscript appears will automatically be sent to the author.

SUBSCRIPTION and MEMBERSHIP information will be found on the back cover.

TEXAS MATHEMATICS TEACHER

TEXAS COUNCIL OF TEACHERS OF MATHEMATICS

Affiliated with the National Council of Teachers of Mathematics

May 1990

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President's Message

MONEY! MONEY! EVERYWHERE! "BUT NONE FOR EDUCATION," SAYS CLEMENTS.

Yes, Texas teachers must again make do with less. We are told that we must increase the performance of our students; keep more students in school; graduate more students in the future; yet we are to do all of this with less monies than we have had in the past. "Just how," you should ask? YES, this is a very good question and I just wish I had the answer. I would bottle it, HAWK it like SAXON, sell it as a CURE ALL, and soon retire rich from teaching. WHO CARES TO JOIN ME IN THIS ADVENTURE TO NEVER, NEVER LAND?

To help offset some of the above mentioned shortcomings, please keep the following ideas and topics in front of you during the summer break. (This might be your last summer to do so if year around schools are voted in by our leaders.) Give them some deep thought and serious considerations.

1. We have a very workable state mathematics curriculum (yes, it might be improved). The ESSENTIAL ELEMENTS are only a minimum, not a maximum for the teaching of mathematics. You can go far beyond the elements if you so desire, just as long as you cover them.

2. MATHEMATICS FRAMEWORK from TEA is a great set of guidelines for course sequencing. If every school district and school would follow the outlines and adopt some type of placement policy, many of our jobs as teachers and leaders would be so easy. We would know how to place students when they transfer to our school.

3. We have the NATIONAL STANDARDS. These guidelines are not NEW curriculum, but just a new approach (to many, not all) to the teaching of mathematics. Let's learn to use these STANDARDS to the fullest.

4. EVERYBODY COUNTS is a great set of guidelines to follow in teaching mathematics. We just need more teachers who will read and

follow these suggestions. YES, we do need more students who will listen.

5. RESHAPING SCHOOL MATHEMATICS - a Philosophy and Framework for Curriculum. A publication from the Mathematical Sciences Education Board, National Research Council. YES, I know that most of you have not read this yet. Your supervisor should have a copy, or may obtain a copy for a small charge. Try to read this as soon as you can. It too is very helpful.

6. NEW TEXTBOOKS for algebra, geometry and trigonometry. We did have a choice this year for some changes in our mathematics programs. I just hope that each district was able to take advantage of the best while the opportunity was there.

7. New textbooks for GRADES K-8 will be chosen during the next year by all districts in Texas. Let's get our input into the proper channels NOW. Let's not wait and then say, "I did not have any say in the selection process." You always do, if you speak before, not after the fact. If you do not know where to speak or to whom to say what, then drop me a note with your suggestions. I will place them in the proper channels.

Keep up the good work. I know that most of you are dedicated, hard-working individuals. It has been my pleasure to meet you, serve with you, work for you and be your President for the past two years. It has been great. I have enjoyed the friendships, fun, fellowship and hard work that the job has provided me. I shall cherish it always. Again, THANKS FOR LETTING ME SERVE YOU.

OTTO W. BIELSS, JR.
President, TCTM

*******PLEASE NOTE*******

The officers and the Board of Directors will meet on Monday, August 6, 1990 at 7:00 p.m. for bringing together our work and constitution. If other sessions are necessary, we will set them up at that time. THANKS! A letter will come later to let you know the location.

TCTM BREAKFAST AND BUSINESS MEETING

It has become an annual event. Meet with the group for breakfast and business at 7:00 a.m. on Wednesday morning, August 8, 1990. See the CAMT bulletin for name and location of the breakfast.

Mail reservations by June 15, 1990. The event is for members only and will be by reservation only. This year a small charge of $3.00 per person will be charged. This will help verify that you have made reservations and also to help you keep your appointments.

You will receive a ticket by mail. Please give home mailing addresses only. Do not give school address. We lost a number of reservations last year and school address was the number one killer.

MAIL BY JUNE 15, 1990 TO: **Maggie Dement, Treasurer**
4622 Pine Street
Bellaire, Texas 77401

Name ______________________________
Please Print

Home Address ______________________________

____

Home Telephone (____) ______________________________

Area of Interest (please check one) ___K-5 ___6-8 ___9-12

Local Council Name ______________________________

Enclosed is my $3.00. Please reserve me a place at the TCTM Breakfast and Business Meeting on August 8, 1990.

____
Signature

*******Remember Board Meeting on August 6, 1990 at 7:00 p.m.*******

HELP NEEDED AT CAMT REGISTRATION

TCTM has the very important responsibility at the Conference for the Advancement of Mathematics Teaching (CAMT) for on-site registration. During the entire conference the registration table is to be open. Working there is exciting and fun. You meet such wonderful and cooperative people. Every participant at CAMT will want to stop by and greet their friends. This conference is for teachers of mathematics, so please contribute some of your time in helping others and to help make the conference a success. The more volunteers we have, the fewer hours anyone will need to spend at the table and we do want everyone to enjoy some of the many wonderful sessions that will be provided. We hope to have a very large crowd in DALLAS and to have just as successful a conference as we have had the past several summers. We (Bill Hopkins) solved most of the registration problems last year. It will be better this year and we hope it will go just as smooth as it went in San Antonio last summer. THE TABLE MUST BE STAFFED AT ALL TIMES. WE DO NEED SOME HELP TO CLOSE OUT ON THURSDAY.

We also need help at the NCTM sales booth (table). Bill does such a wonderful job, but he needs your help. Let's give him some of our time.

Use this form to volunteer to work during CAMT.

MAIL BY JUNE 15, 1990 TO: **Maggie Dement, Treasurer**
4622 Pine Street
Bellaire, Texas 77401

Time(s) I will help ______________________

Time(s) that I cannot help ______________________

Times do not matter, assign me whenever you most need me __________

Name (please print) ______________________
Home Address ______________________
City, State, Zip ______________________
Home Telephone (   ) ______________________

____
Signature

****Remember Board of Directors Meeting on August 6, 1990 at 7:00 p.m.****

INTRODUCING THE POINT OF DIVISION FORMULAS FOR A LINE SEGMENT

David J. Cochener

Department of Mathematics and Computer Science
Austin Peay State University
Clarksville, Tennessee

The purpose of this note is to present a convenient and natural method for introducing the general point of division formulas for a line segment which greatly facilitates their use.

The first point of division formulas a student sees are the midpoint formulas. He/she learns that the coordinates of the midpoint M(x,y) of the segment AB are given by $x = (x_1 + x_2)/2$ and $y = (y_1 + y_2)/2$ where $A(x_1,y_1)$ and $B(x_2,y_2)$ are the endpoints of the segment. The most effective way for the student to remember these formulas is to realize that the coordinates of the midpoint are just the averages of the corresponding coordinates of the endpoints. In this way he/she remembers to add and divide by two rather than subtracting as in the distance formula (which he/she has likely just seen).

The midpoint formulas are then generalized to find the coordinates of a point P(x,y) on segment AB which divides the segment in the ratio $r_1:r_2 = |AP|:|PB|$, where r_1 and r_2 are positive numbers, usually positive integers, and where $|AP|$ denotes the length of segment AP. If A has coordinates (x_1,y_1) and B has coordinates (x_2,y_2), then by using similar triangles or parallel projections on the coordinates axes it is established that $(x - x_1)/(x_2 - x) = r_1/r_2$ and $(y - y_1)/(y_2 - y) = r_1/r_2$. (See Figure 1.) Thus the values for x and y are given by $x = (r_2x_1 + r_1x_2)/(r_1 + r_2)$ and

$y = (r_2y_1 + r_1y_2)/(r_1 + r_2)$. Clearly the midpoint is the special case where $r_1 = r_2 = 1$.

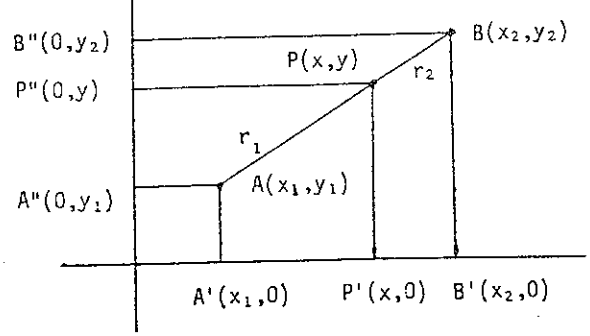

Figure 1.

Not only do the formulas generalize, but the interpretation of the formulas does as well. The value of x given above represents the weighted average of the x-coordinates of the endpoints where x_1 has weight r_2 and x_2 has weight r_1. Similarly y is the weighted average of the y-coordinates of the endpoints. Noticing this gives the student an easy and natural way to remember the formulas. In fact, the correct assignment of weights can be found from the conditions of the problem just by considering whether P is closer to A or to B. The coordinates of the point to which P is closer will receive the greater weight.

EXAMPLE. Let A(1,-3) and B(7,2). Find the coordinates of the point P(x,y) on segment AB so that P is three times as far from A as from B.

SOLUTION. It is helpful to draw and label a rough diagram. (See Figure 2.) Let d = |PB|. Then |AP|:|PB| = 3d:d = 3:1 and P divides the segment in the ratio 3:1. Since P is nearer to B the coordinates of B receive more "weight" and so

$$x = [\underline{3}(7) + \underline{1}(1)]/(3 + 1) = 22/4$$

and

$$y = [\underline{3}(2) + \underline{1}(-3)]/(3 + 1) = 3/4.$$

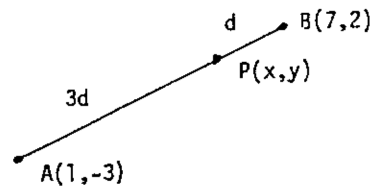

Figure 2.

An easy way for the student to verify his/her results is to project A, B and P onto the coordinate axes and use the fact that parallel projections preserve ratios. For the x-coordinate the projections of A, B and P on the x-axis are A'(1,0), P'(22/4,0) and B'(7,0). We find that |A'P'|:|P'B'| = (22/4 - 1)/(7 - 22/4) = 18:6 = 3:1. A similar check for y-coordinates yields projections A"(0,-3), P"(0,3/4) and B"(0,2) so that |A"P"|:|P"B"| = [3/4 - (-3)]/(2 - 3/4) = 15:5 = 3:1. In general this is much easier than applying the distance formula to the segments AB, AP and PB.

At this point the student may be introduced to the fact that a line segment AB may be "divided" externally by a point P. With the use of directed distances and negative ratios he/she is shown that the previous formulas are still valid. There is really no need for this as any of these problems can be viewed as an internal division problem where one of the given points does the dividing.

<u>EXAMPLE</u>. Again let A(1,-3) and B(7,2). Find the coordinates of the point P(x,y) on the line AB, but not on segment AB, such that P is three times as far from A as from B.

<u>SOLUTION</u>. Draw and label a diagram. (See Figure 3.) In this case |AP|:|BP| = 3:1 so B divides segment AP in the ratio 2:1 = |AB|:|BP|. Thus the "weights" are 2 and 1 and since B is closer to P, the coordinates of P receive greater "weight." Therefore we have

$$7 = [\underline{2}(x) = \underline{1}(1)]/(2 + 1) \text{ or } x = 10$$

and

$$2 = [\underline{2}(y) + \underline{1}(-3)]/(2 + 1) \text{ or } y = 9/2.$$

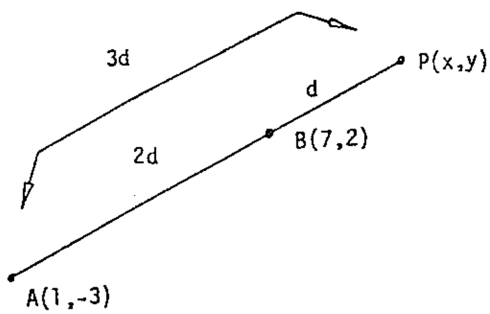

Figure 3.

Again the student can use parallel projection to check his/her work by verifying that the proper ratios hold between the x-coordinates and y-coordinates. The advantage of the approach given in the example is that all mention of directed distances and negative ratios is avoided on a first exposure.

In introducing any mathematical concept it is a good idea, if possible, to provide the student with a physical interpretation of the concept. Such a physical interpretation for the point of division formulas is provided by the idea of a center of mass. Consider the following situation. A mass of 2 units is located at the point A(-5,1) and a mass of 3 units is located at the point B(4,1). Then the center of mass or "balance point" for these two masses is located at the point P(x,1) where $x = [2(-5) + 3(4)]/(2+3) = 2/5$. You can verify that this is the point that is 3/5 of the way from A to B or, equivalently, the point that divides the segment AB in the ratio 3:2 = |AP|:|PB|. Note that the point of division P is closer to the point having the greater mass.

In general, if $A(x_1,y)$ and $B(x_2,y)$ are points on a horizontal line, then the point P(x,y) that divides the segment AB in the ratio $r_1:r_2 = |AP|:|PB|$ has x-coordinate equal to $(r_2x_1 + r_1x_2)/(r_1 + r_2)$ and this point is the center of mass of the system with a mass of r_2 units at A and a mass of r_1 units at B. The same argument also applies to vertical lines.

If we now let $A(x_1,y_1)$ and $B(x_2,y_2)$ be any two points in the plane and if we assume that a mass of r_2 units is located at A and a mass of r_1, units is located at B, then the center of mass for the two points will be at the point P(x,y) where $x = (r_2x_1 + r_1x_2)/(r_1 + r_2)$ and $y = (r_2y_1 + r_1y_2)/(r_1 + r_2)$. These are precisely the coordinates of the point that divides segment AB in the ratio $r_1{:}r_2 = |AP|{:}|PB|$.

The idea of a center of mass is an important one and can be generalized to masses located at any finite set of points or, with the aid of calculus, to a flat plate in the xy-plane whose mass density is a continuous function of x and y.

In the presentation of any new topic it is always helpful to relate it to topics with which the student has some familiarity. By presenting the point of division formulas as averages, and showing their interpretation as a center of mass, the student is given a convenient and consistent way to remember and apply the formulas while relating them to ideas which are important in their own right.

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USING COMPUTER ALGEBRA SYSTEMS FOR INDETERMINATE FORMS

John Mathews

Professor of Mathematics
California State University, Fullerton
Fullerton, California

Computer algebra systems such as MACSYMA, Mathematica, MAPLE and muMATH can be used as an assistant for performing symbol manipulations in algebra and calculus (see references [1-9]). High school and undergraduate students in mathematics can benefit from these interactive software packages which perform the necessary complicated manipulations. The theme of this article is to show how a computer algebra system can be used to illustrate the mathematical theory for finding limits which have either the indeterminate from "0/0" or "∞/∞." The computer algebra system muMATH used in this article, is available for IBM compatible computers. A version, muMATH-80, has been released which runs on the APPLE II family of computers.

In calculus we teach techniques for finding the limit of a quotient

(1) $$\lim_{x \to a} \frac{f(x)}{g(x)}$$

For the trivial case when $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, and the latter is not zero, the limit is easy to determine and its value is f(a)/g(a). This article is concerned with quotients which have the indeterminate form "0/0" or "∞/∞".

QUOTIENT OF POLYNOMIALS.

Assume that the value a is finite and that both f(x) and g(x) are polynomials of degree m and n respectively and that f(a) = 0 and g(a) = 0. Then there exist positive integers j and k so that f and g have zeros of order j and k, respectively. To be precise, we can write

(2) $f(x) = (x-a)^j r(x)$ and $g(x) = (x-a)^k s(x)$

where r(x) and s(x) are polynomials of degree m-j and n-k, respectively, and r(a) ≠ 0 and s(a) ≠ 0. In view of (2) we can simplify problem (1) as follows:

$$(3) \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{r(a)}{s(a)} \lim_{x \to a} (x-a)^{j-k}.$$

There are three cases to consider;

Case (i). When j > k the limit is 0.
Case (ii). When j = k the limit is r(a)/s(a).
Case (iii). When j < k the limit is undefined.

It is clear to see that the problem is solved if we can factor both f and g in the form (2); however, this can be tedious. This is where a computer algebra system can be of assistance. There is a built-in muMATH command PQUOT(dividend,divisor,subexpr), which returns the polynomial quotient evaluated at x=a. It can be used to create the muMATH subroutine that we call PRULE(N,D,A), which will recursively factor out the common term (x-a) until the determination of the limit is made. The program listing for PRULE is given at the end of the article. Let us illustrate its utility.

EXAMPLE 1. Find $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$.

SOLUTION. First, place the formulas for the numerator and denominator in the variables N and D, respectively.

```
? N: X^3 - 8;     and    ? D: X^2 - 4;
@: -8 + X^3              @: -4 + X^2
```

Next, type the new muMATH command

```
? PRULE(N,D,2);
```

The response by muMATH will be the following printout which tells about each step being performed.

```
Taking the limit as X--> 2

  Numerator = -8 + X^3
Denominator = -4 + X^2

This is a case of '0/0'.
Remove the common factor -2 + X from numerator and denominator.

Taking the limit as X--> 2

  Numerator = 4 + 2 X + X^2
Denominator = 2 + X

      Limit = 12 / 4 = 3

@: DONE
```

Therefore, muMATH has found that

$$\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)}{(x-2)(x+2)} = \lim_{x \to 2} \frac{x^2 + 2x + 4}{x + 2} = 3 .$$

EXAMPLE 2. Find

$$\lim_{x \to 2} \frac{x^4 - 2x^3 - 7x^2 + 20x - 12}{x^4 - 9x^2 + 4x + 12} .$$

SOLUTION. First, put the formulas in the variables N and D

?N: X^4 - 2X^3 - 7X^2 + 20X -12; and ?D: X^4 - 9X^2 + 4X +12;

respectively. Then ask muMATH for the results.

```
? PRULE (N,D,2);

Taking the limit as X-->2

  Numerator = -12 + 20 X - 7 X^2 - 2 X^3 + X^4
Denominator = 12 + 4 X - 9 X^2 + X^4

This is a case of '0/0'.
Remove the common factor -2 + X from numerator and denominator.

Taking the limit as X--> 2

  Numerator = 6 - 7 X + X^3
Denominator = -6 - 5 X + 2 X^2 + X^3

This is a case of '0/0'.
Remove the common factor -2 + X from numerator and denominator.

Taking the limit as X--> 2

  Numerator = -3 + 2 X + X^2
Denominator = 3 + 4 X + X^2

      Limit = 5 / 15 = 1/3

@: DONE
```

Thus, muMATH has found that

$$\lim_{x \to 2} \frac{x^4 - 2x^3 - 7x^2 + 20x - 12}{x^4 - 9x^2 + 4x + 12} = \frac{1}{3}.$$

L'Hôpital's Rule.

Assume that the value a is finite or infinite and both f(x) and g(x) differentiable functions, and that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \quad (\text{or } \infty) \quad \text{and that}$$

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} \quad \text{exists. Then}$$

(4) $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} .$$

The built-in muMATH function DIF(expression, variable) can be used to carry out the differentiation task. Then a logical check to determine whether the case "0/0" or "∞/∞" occurs and whether the process needs to be used recursively can be made by muMATH. The details are carried out by a program called LRULE(N,D,A), which is listed at the end of the article.

EXAMPLE 3. Find $$\lim_{x \to 0} \frac{e^x - \cos x - \sin x}{x \sin x} .$$

SOLUTION. Place the formulas for the numerator and denominator in the variables N and D with the commands ? N: #E^X - COS X - SIN X; and ? D: X SIN X; Then issue the new muMATH command:

? LRULE(N,D,O);

The printed response by muMATH will tell about each step.

Taking the limit as X--> 0

Numerator = #E^X - COS X - SIN X
Denominator = X SIN X

This is a case of '0/0'.
Apply L'Hôpital's rule.

Taking the limit as X--> 0

Numerator = #E^X - COS X + SIN X
Denominator = X COS X + SIN X

This is a case of '0/0'.
Apply L'Hôpital's rule.

Taking the limit as X--> 0

Numerator = #E^x + COS X + SIN X
Denominator = -X SIN X + 2 COS X

Limit = 2 / 2 = 1

@: DONE

Therefore, muMATH has found that

$$\lim_{x \to 0} \frac{e^x - \cos x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{e^x + \sin x - \cos x}{x \cos x + \sin x}$$

$$= \lim_{x \to 0} \frac{e^x + \cos x + \sin x}{2 \cos x - x \sin x}$$

$$= 1 .$$

EXAMPLE 4. Find $\lim_{x \to \infty} \frac{4x^3 - x^2 + x - 6}{3x^3 + x^2 + 4x + 1}$.

SOLUTION. Place the formulas in the variables N and D with the commands ? N: 4 X^3 - X^2 + X - 6; and ? D: 3 X^3 + X^2 + 4 X + 1;

respectively. muMATH uses the symbol PINF for ∞ and the solution is found by typing the command ? LRULE(N,D,PINF); The printed response is as follows:

Taking the limit as X--> PINF

Numerator = -6 + X - X^2 + 4 X^3
Denominator = 1 + 4 X + X^2 + 3 X^3

This is a case of '∞/∞'.
Apply L'Hôpital's rule.

Taking the limit as X--> PINF

Numerator = 1 - 2 X + 12 X^2
Denominator = 4 + 2 X + 9 X^2

This is a case of '∞/∞'.
Apply L'Hôpital's rule.

Taking the limit as X--> PINF

Numerator = -2 + 24 X
Denominator = 2 + 18 X

This is a case of '∞/∞'.
Apply L'Hôpital's rule.

Taking the limit as X--> PINF

Numerator = 24
Denominator = 18

Limit = 24 / 18 = 4/3

@: DONE

Thus, muMATH has found that

$$\lim_{x \to \infty} \frac{4x^3 - x^2 + x - 6}{3x^3 + x^2 + 4x + 1} = \frac{4}{3}.$$

PROGRAM LISTINGS.

The following muMATH programs carry out the steps illustrated in the article. They can be typed in the computer using the muMATH editor or they can be placed in a text file named RULES.ALG and then read with the muMATH command RDS('RULES,ALG) .

```
FUNCTION PRULE (N, D, A),
   NEWLINE (1), PRINT ("Taking the limit as X--> "),
   PRTMATH (A), NEWLINE (2),
   PRINT (" Numerator = "), PRTMATH (N),
   NEWLINE (1), PRINT ("Denominator = "),
   PRTMATH (D), NEWLINE (1),
   N0: LIM (N, X, A), D0: LIM (D, X, A),
   N: PQUOT (N, X - A, X), D: PQUOT (D, X - A, X),
   WHEN NOT (D0 = 0 OR D0 = PZERO OR D0 = MZERO),
      NEWLINE (1), PRINT ("  Limit =  "),
      PRTMATH (N0), PRINT (" / "),
      PRTMATH (D0), PRINT (" = "),
      PRTMATH (N0/D0), NEWLINE (2), DONE
      EXIT,
   WHEN (N0 = 0 OR N0 = PZERO OR N0 = MZERO)
      AND (D0 = 0 OR D0 = PZERO OR D0 = MZERO),
      NEWLINE (1), PRINT ("This is a case of '0/0'."),
      NEWLINE (1), PRINT ("Remove the common factor "),
      PRTMATH (X - A), NEWLINE (1),
      PRINT ("from numerator and denominator."),
      NEWLINE (1),
      PRULE (N, D, A)
      EXIT,
   NEWLINE (1), PRINT ("This is a case of "),
   PRTMATH (N0), PRINT (" / "),
   PRTMATH (D0), NEWLINE (2),
   UNDEFINED
ENDFUN$

FUNCTION LRULE (N, D, A),
   NEWLINE (1), PRINT ("Taking the limit as X--> "),
   PRTMATH (A), NEWLINE (2),
   PRINT (" Numerator = "), PRTMATH (N),
   NEWLINE (1), PRINT ("Denominator = "),
```

```
PRTMATH (D), NEWLINE (1),
N0: LIM (N, X, A), D0: LIM (D, X, A),
N: DIF (N, X), D: DIF (D, X),
WHEN NOT (N0 = PINF OR N0 = MINF OR N0 = ? OR
   D0 = 0 OR D0 = PZERO OR D0 = MZERO OR D0 = ?),
   NEWLINE (1), PRINT ("  Limit =  "),
   PRTMATH (N0), PRINT (" / "),
   PRTMATH (D0), PRINT (" = "),
   PRTMATH (N0/D0), NEWLINE (2), DONE
   EXIT,

WHEN (N0 = 0 OR N0 = PZERO OR N0 = MZERO)
   AND (D0 = 0 OR D0 = PZERO OR D0 = MZER0),
   NEWLINE (1), PRINT ("This is a case of "0/0."),
   NEWLINE (1), PRINT ("Apply L'Hôpital's rule."),
   NEWLINE (1), LRULE (N, D, A)
   EXIT,
WHEN (N0 = PINF OR N0 = MINF) AND (D0 = PINF OR D0
        = MINF),
   NEWLINE (1), PRINT ("This is a case of '∞/∞'."),
   NEWLINE (1), PRINT ("Apply L'Hôpital's rule."),
   NEWLINE (1), LRULE (N, D, A)
   EXIT,
NEWLINE (1), PRINT ("This is a case of "),
PRTMATH (N0), PRINT (" / "),
PRTMATH (D0), NEWLINE (2),
UNDEFINED
ENDFUN$
```

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REFEREES WANTED

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INFORMAL ASSESSMENT OF COMPUTATIONAL ERRORS

Evelyn M. VanDevender

Department of Elementary/Early Childhood Education
University of South Alabama
Mobile, Alabama

Effective teaching in mathematics, as in any other subject, depends upon assessment of students' learning needs. The "Curriculum and Evaluation Standards for School Mathematics" published by the National Council of Teachers of Mathematics (1989) provides directives for teaching and evaluating mathematics in the future. These directives include making conjunctures, verifying answers, and communicating results, all of which should be incorporated into the informal assessment program. To help implement these evaluation standards, different assessment techniques are required.

Research findings indicate that frequent and systematic evaluation is essential for effective elementary mathematics instruction (Ashlock, 1986; Bennett, 1986; Reisman, 1982; Underhill, Uprichard & Heddens, 1979). Therefore, valid and reliable assessment techniques are critical for good mathematics instruction.

Commercialized test scores are often used to measure student achievement in mathematics. These scores may or may not reflect the exact area of need for each student. In order to provide a more comprehensive mathematical assessment program, informal assessments are needed to pinpoint precise strengths and weaknesses of each student's mathematical competencies. Four often overlooked informal assessment techniques are described to help teachers detect student mistakes, increase motivation, and improve mathematics achievement.

MENTAL EXERCISES

Mental exercises can be used to see how quick, alert, and accurate students are in their mathematical thinking. These exercises can heighten students' awareness of the usefulness of mathematics and can be performed without paper and pencil. Cruikshank and Sheffield (1988) believe that mental calculations and estimation should be built into every aspect of mathematics teaching.

The type of exercise will vary according to the student's ability level. In this assessment the student is asked to give the answer to an orally presented exercise. For example, the following questions would be appropriate to evaluate kindergarten, first, or second grade students' estimation skills:

How tall are you?
How tall am I?
How much does a new doll cost?
How much does a new shirt cost?

Some mental computation questions for this same ability level might include:

When we count, what number comes after four?
What number comes before ten?
What is six minus four?
Name some things that come in twos.

Teachers can help analyze their students' thinking processes by asking, "How did you get that answer?" or "Why do you think that answer is correct?" This helps students sort through their thinking to organize and clarify their mathematical processes. Prompt feedback from teachers on these mental exercises will help improve student motivation and achievement.

CONCEPTS OF OPERATIONS

Concepts of the four basic operations (addition, subtraction, multiplication, and division) can be used to see how well students understand the operation. Numerous studies (Baratta-Lorton, 1978; Bruner, 1983; Piaget, 1952; Tunis, 1986) support the belief that students should represent mathematical concepts with physical objects (manipulatives). In this assessment the student is asked to tell a real-life story and model it with a manipulative to show the meaning of the operation.

A possible student response for addition is given in Figure 1. This procedure is followed for each of the other operations shown in Figure 1. If students can generate a story for the operation and model it with manipulatives, they have a good concept of the operation.

Concepts of Operations

1. Addition: Real-life story
   - Model o o + o o o
   - Symbols 2 + 3 = [5]
2. Subtraction: Real-life story ______
   - Model ______
   - Symbols 7 - 3 = [ ]
3. Multiplication: Real-life story ______
   - Model ______
   - Symbols 4 x 3 = [ ]
4. Division: Real-life story ______
   - Model ______
   - Symbols 12 ÷ 4 = [ ]

Figure 1.

WORK ALOUD

The work aloud procedure is one of the most valuable tools for identifying errors. Kennedy and Tipps (1988) recommend asking students to "think aloud" as they work to identify their thought processes. It reveals a student's thinking patterns, the nature of specific understandings, and what strategies the student uses for solving problems. In this procedure the teacher asks the student to "think aloud" as he/she works the exercise. Faulty concepts will come to the surface almost immediately. The teacher must be sure the student verbalizes every step in the problem. An example is demonstrated in the following algorithm:

$$\begin{array}{r} 16 \\ \underline{\times 3} \\ 68 \end{array}$$

In working the algorithm aloud, the student says, "Three times six is eighteen; put down eight and carry one; one plus one is two and two times three is six." The student erroneously added the one to the one in the tens place before multiplying by three. By using the work aloud procedure, the teacher can immediately pinpoint the exact error from the student's incorrect process and plan corrective instruction.

A variety of tasks using the work aloud procedure should be used so the teacher can better understand what the student knows as well as what the student can do. It also provides students with an effective strategy for learning to assess their own strengths and weaknesses in mathematics.

SENTENCE BUILDING

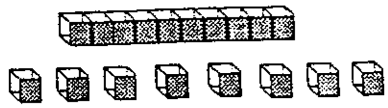
The sentence building assessment helps teachers discover their students' understanding of regrouping. Cruikshank and Sheffield (1988) advise teachers to use concrete materials, such as the base ten blocks, and to have students record the regrouping as they trade blocks. In this assessment the student is asked to use the Multibase Arithmetic Blocks to show with the fewest pieces of wood the meaning of the addition or subtraction algorithm. (see Figure 2).

Sentence Building

	Show me with fewest pieces of wood (Multibase Arithmetic Blocks)	REGROUP	RECOGNIZE EQUIV.	STATE EQUIV.	COMPUTE (Written Record)
ADDITION	18 + 4	√	√	√	$\begin{array}{r} {\scriptstyle 1} \\ 18 \\ +\ 4 \\ \hline 22 \end{array}$
	25 + 8				$\begin{array}{r} 25 \\ +\ 8 \\ \hline \end{array}$
SUBTRACTION	13 - 6				$\begin{array}{r} 13 \\ -\ 6 \\ \hline \end{array}$
	34 - 18				$\begin{array}{r} 34 \\ -\ 18 \\ \hline \end{array}$

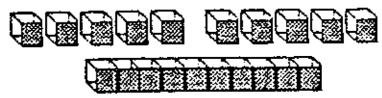
Figure 2.

For example, in the first algorithm (18 + 4) the student illustrates 18 with one long and eight units

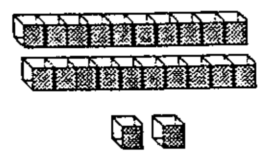


He/she takes four more units. 

Now he/she must trade ten units for one long to show the answer with the fewest pieces of wood.



Now the student has two longs and two units.



In the third column of Figure 2 the student verbalizes what he/she did with the Multibase Arithmetic Blocks, and in column four he/she states the equivalence. If the student can do this, the teacher checks the columns as shown on Figure 2 with the first addition sentence. In the last column, the written record of what was done with the blocks is shown on paper. The student is asked if he/she sees the connection between what was done with the blocks and what occurred on paper when the eight and four were regrouped into two tens and two ones. The other algorithms in Figure 2 are completed following the same procedure.

SUMMARY

The four assessment techniques presented in this article can be used in almost any elementary classroom setting. They will help teachers correct their students at each stage of learning to improve instruction. Teachers can design and carry-out their own assessments in their own classroom. These assessments are valuable, inexpensive, and easily applied procedures for discovering the reasons students make mistakes.

Since these assessments are used during actual teaching, they take little additional teacher time, but they make the student's learning more effective by providing immediate feedback and by correcting misconceptions before they are firmly fixed in the student's mind. The teaching, diagnosing, reteaching, extending should be an ongoing daily process.

The author has used these informal assessments in a University Mathematics Center for the last seven years. More planning and instructional time are needed to implement these procedures; however, it has been a joy to see the dramatic difference that is occurring in students lives because of these assessments. Teachers and parents of these students tell you their achievement and grades have improved. More important, the student themselves tell you they now "like math."

Elementary mathematics teachers should use a wider variety of assessment techniques to identify exact individual student's instructional needs. Only in this way will the teacher develop a program that will provide students with a firm mathematical foundation and help them avoid further problems as they progress through school.

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TRIANGLES WITH RATIONAL SINES

Janice W. Schwartzman

University High School
Irvine, California

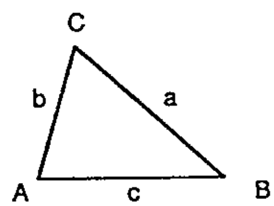
Harris S. Schultz

Department of Mathematics
California State University, Fullerton
Fullerton, California

In [1], McLean notes the following interesting fact:

(1) If the sines of all the angles of a triangle are rational, then so are the cosines.

This can be proved as follows. If in ▲ABC, we have sin A, sin B, and sin C all rational, define



The length c of side $\overline{AB}$ to be 1 unit. Then, the side lengths a and b must also be rational since, by the law of sines,

(2) $$a = \frac{\sin A}{\sin C} \text{ and } b = \frac{\sin B}{\sin C}.$$

Using the law of cosines,

$$\text{(3)} \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Therefore, cos A is rational. Similarly, cos B and cos C are rational.

Surprisingly, the converse is false. As pointed out by McLean, this is illustrated by observing that, for any equilateral triangle, the cosine of each angle is the rational number 1/2 while the sine of each angle is the irrational number $\sqrt{3}/2$.

In reflecting upon (1), we were curious to know which triangles actually have the property that the sines and cosines of all the angles are rational. Certainly, no triangle with either a 30° or 150° angle has this property since cos 30° and cos 150° are irrational. However, any right triangle with rational sides does have this property. How about non-right triangles? We shall now show how examples can be constructed using pairs of Pythagorean triples.

Let u,v,w and x,y,z be Pythagorean triples (perhaps the same). That is, suppose u, v, w, x, y, and z are natural numbers such that

$$\text{(4)} \qquad u^2 + v^2 = w^2 \text{ and } x^2 + y^2 = z^2.$$

Observing that $u<w$ and $x<z$, we define ▲ABC so that

$$\text{(5)} \qquad A = \sin^{-1}\frac{U}{W},\ B = \sin^{-1}\frac{X}{Z},\ C = 180-(A+B)$$

(angles A and B are acute by definition of the inverse sine function). Then $\sin A = u/w$, $\sin B = x/z$, $\cos A = v/w$, and $\cos B = y/z$. Applying the sum-difference formulas, we have

$$\text{(6)} \qquad \sin C = \sin[180-(A+B)] = \sin(A+B) = (uy+xv)/(wz)$$

and

$$\text{(7)} \qquad \cos C = \cos[180-(A+B)] = -\cos(A+B) = (ux-vy)/(wz).$$

Note that the sines and cosines of all the angles are rational and that

(8) $$\frac{a}{b} = \frac{\sin A}{\sin B} = \frac{uz}{wx} \text{ and } \frac{a}{c} = \frac{\sin A}{\sin C} = \frac{uz}{uy + xv}.$$

Now, by (8), any ▲A'B'C' for which a'/b' = (uz)/(wx) and a'/c' = (uz)/(uy + xv) is similar to ▲ABC and, therefore, the sines and cosines of all its angles are rational. The following table gives examples of triangles with side lengths uz, wx, and uy + xv.

u	*v*	*w*	*x*	*y*	*z*	*uz*	*wx*	*uy+xv*
4	3	5	4	3	5	20	20	24
4	3	5	12	5	13	52	60	56
3	4	5	12	5	13	39	60	63
4	3	5	5	12	13	52	25	63
3	4	5	3	4	5	15	15	24
12	5	13	24	7	25	300	312	204

Notice that a triangle having side lengths 5,5 and 6 is similar to the first triangle in the table and, therefore, the sines and cosines of all its angles are rational. Likewise, using the second example, a triangle having side lengths 13, 15, and 14 possesses this property. Observe that, since cos C = (ux - vy)/(wz), angle C is obtuse if and only if ux < vy.

The proportions a/b = (uz)/(wx) and a/c = (uz)/(uy + xv) in (8), where u,v,w and x,y,z are Pythagorean triples, provide not only sufficient, but also necessary conditions for the sines of the angles of ▲ABC to be rational. Indeed, if sin A, sin B, and sin C are rational, then, by (1), so are cos A, cos B, and cos C. Therefore, each of the numbers sin A, cos A, sin B, and cos B can be expressed as a quotient of natural numbers. That is, we can write sin A = u/w, cos A = v/w, sin B = x/z, and cos B = y/z, where u, v, w, x, y, and z are natural numbers. The identities $\sin^2 A + \cos^2 A = 1$ and $\sin^2 B + \cos^2 B = 1$ tell us that

(9) $$u^2 + v^2 = w^2 \text{ and } x^2 + y^2 = z^2.$$

Reasoning as in (6) and (8), we see that sin C = (uy + xv)/(wz) and, finally, a/b = (uz)/(wx) and a/c = (uz)/(uy + xv).

Reference

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CONFERENCE FOR THE ADVANCEMENT OF MATHEMATICS TEACHING

CAMT GOES TO DALLAS

MATHEMATICAL LITERACY:
ACCESS TO THE FUTURE

August 7, 8, and 9, 1990, Hyatt Regency DFW
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Preconference
The Goal -
Mathematical Literacy
For All Students
August 6, 1990

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George H. Willson, Editor
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Teachers of Mathematics
P. O. Box 13857
University of North Texas
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A Publication for Elementary School Mathematics

Volume II, No. 3 May, 1990

A Section of Texas Council of Teachers of Mathematics

CAMT GOES TO DALLAS

Elementary Teachers:

CAMT will be held in Dallas this year on August 7, 8, and 9, 1990 at the Hyatt Regency, DFW. If you are flying in, you will be pleased to know that the conference hotel is at the DFW Airport. The theme of the conference is **Mathematical Literacy for All Students.** There will be many sessions of special interest for elementary and primary teachers teaching mathematics.

STEAM will hold its annual "wing-ding" for elementary teachers of mathematics with surprise drawings!

There will be a pre-conference on August 6. You will be receiving more information on this in the conference program that should be arriving soon.

Writing in Mathematics notes

Within the newsletter you will find an article entitled **STUDENT PORTFOLIOS.** This is a page from the booklet, Assessment Alternatives in Mathematics. Copies can be obtained by writing:

EQUALS
Lawrence Hall of Science
University of California
Berkeley, CA 94720
Attn: Assessment Booklet

- - - - - - - - - - - - - - - - - -

Marilyn Burns Educational Association has a workshop entitled **Writing in Mathematics** that is very helpful in making the mathematics and writing connection. These workshops are held throughout the year.

WRITING IN MATHEMATICS

Cathy's Corner

Communication in Mathematics — A Critical Link

Communication is one of the K-12 threads in the National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards for School Mathematics* . At the elementary level, this involves basic representations of problems and everyday situations into mathematical ideas, student explanations of how a problem was solved, development of mathematical language, and the use of written or oral words to communicate mathematical ideas.

One of the most exciting mathematics experiences I have ever seen took place in an elementary classroom during a language arts lesson. The teacher was working with students on making their own books, and the day I visited they were working on books about multiplication. In previous days' discussions, the students had brainstormed everything they knew about multiplication — what it meant, how it was used, problems where it applied, how you could write it, and so on. They were busy putting those thoughts into their little newsprint books, and they were loving every minute of it. The level of mathematics in the students' discussions was far beyond what we normally see in elementary mathematics. Those students were making generalizations about multiplication that would serve them for the rest of their lives. When they later come across a situation that calls for multiplication, they will automatically connect that situation to their early experiences where they formed their generalizations about what multiplication is.

Asking a student to explain how she or he arrived at a solution, or having a student keep a journal in mathematics can serve several purposes for both the student and the teacher. For the student, explaining in words what process was used can be an important tool for reflecting on and clarifying the student's own thinking. In the process, powerful generalizations begin to form that allow students to observe what steps have actually occurred and what might work again in the future. The likelihood of the student repeating a successful process is far stronger when the student has articulated that process. In writing a journal entry, the student also clarifies attitudes and feelings about mathematics, a process that can be very useful in heading off future mathematics anxiety.

Meanwhile, the teacher can gain meaningful insights into what the student is thinking by listening to a student's explanation or by reading a journal entry. If the student is using a sound process, whether or not an appropriate solution has been found, the teacher can reinforce that process. Likewise, the teacher can reinforce students who can articulate that their answer makes sense, regardless of the process used. Meanwhile, for students who have some misunderstanding, either small or large, the teacher can provide an intervention that is targeted exactly to the area of difficulty by listening to or looking for where the student got off track. This saves teacher and student time, and can prevent the student from practicing erroneous processes. Teachers often find surprises when they look at or listen to student explanations. What may seem like a totally off-base student response lacking any understanding may in fact represent a far higher level of thinking than the teacher would ever expect. One student, whose answer to a problem was correct, showed a series of steps that appeared to be meaningless, isolated scribblings that clearly showed a lack of understanding of a basic multiplication algorithm. On interviewing the student, it turned out that she had developed a unique and fairly abstract algorithm with a sound mathematical foundation. If the teacher had looked only at the answer, it would have appeared that the student was simply doing what she was taught. If the teacher had looked only at the steps, it would have appeared that the student did not understand the concept. Only by looking at the steps and listening to the student's explanation, did the real level of understanding come forward.

The emphasis on tying together language and mathematics is not new. There is resurgent interest in the integration of the various disciplines at the elementary level these days. Perhaps it is not necessary to design all-encompassing units that try to address every aspect of the curriculum, often trivially. Perhaps it is more appropriate to look at connecting each subject area to others in increasingly sophisticated and extensive ways that maintain the integrity and natural excitement of the subjects themselves. Teaching for communication in mathematics not only serves this purpose, but it also fosters mathematical understanding in the process.

Cathy Seeley, Director of Mathematics, TX Education Agency

Reference: National Council of Teachers of Mathematics, *Curriculum and Evaluation Standards for School Mathematics*, Reston, VA: NCTM, 1989.

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Mathematical Literacy: Access to the Future
August 7-9, 1990
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Preconference
The Goal - Mathematical Literacy for All Students
August 6, 1990

MATH QUOTES

MATHEMATICS MAY BE DEFINED AS THE SUBJECT IN WHICH WE NEVER KNOW WHAT WE ARE TALKING ABOUT, NOR WHETHER WHAT WE SAY IS TRUE. —BERTRAND RUSSELL

THE STUDY OF MATHEMATICS IS APT TO COMMENCE IN DISAPPOINTMENT. THIS GREAT SCIENCE OFTEN ELUDES OUR MENTAL WEAPONS TO GRASP IT. —ALFRED WHITEHEAD

MATHEMATICS TAKES US STILL FURTHER FROM WHAT IS HUMAN INTO THE REGION OF ABSOLUTE NECESSITY, TO WHICH NOT ONLY THE ACTUAL WORLD, BUT EVERY POSSIBLE WORLD, MUST CONFORM. — BERTRAND RUSSEL

THE UNION OF THE MATHEMATICIAN WITH THE POET--- PASSION WITH CORRECTNESS--- IS SURELY THE IDEAL.— WILLIAM JAMES

MATHEMATICS POSSESSES NOT ONLY TRUTH, BUT SUPREME BEAUTY, LIKE THAT OF A SCULPTURE, AND IS CAPABLE OF A PERFECTION SUCH AS ONLY THE GREAT ARTS CAN SHOW. —BERTRAND RUSSEL

THE UNIVERSE IS WRITTEN IN THE LANGUAGE OF MATHEMATICS. —GALILEO

STUDENT PORTFOLIOS

Mathematics Portfolios

Student portfolios are well-known in art and writing, but until now have rarely been used to keep a record of student progress in mathematics. Teachers have always kept folders of student work, but portfolios may now have more focus and be more important for assessment.

What is in a Portfolio?

Teachers and their students should be allowed to choose most of the items to include in portfolios, since it gives a good indication of what is valued. Occasionally it may be desirable, for the sake of comparisons, for some outside agency to ask for inclusion of a certain type of item, but this should be the exception. If possible, teachers and students should be able to present and explain their own portfolios to outside observers.

Putting dates on all papers will become more important. First draft or revised writing should be acceptable, but with a note about which it is. The names of group members should probably be on papers done by a group, or at least an indication that it was group work.

A portfolio might include samples of student-produced:

- written descriptions of the results of practical or mathematical investigations
- pictures and dictated reports from younger students
- extended analyses of problem situations and investigations
- descriptions and diagrams of problem-solving processes
- statistical studies and graphic representations
- reports of investigations of major mathematical ideas such as the relationship between functions, coordinate graphs, arithmetic, algebra, and geometry
- responses to open-ended questions or homework problems
- group reports and photographs of student projects
- copies of awards or prizes
- video, audio, and computer-generated examples of student work
- other material based on project ideas developed with colleagues

Teachers and Portfolios

The definition and evaluation of portfolios are opportunities for teachers to share and learn with peers. Groups of teachers who have reviewed the contents together have found it an exciting and rewarding experience. On page 10 are some examples of teacher comments made during pilot assessments in the spring of 1989. Also, sharing with parents, administrators, and school boards will help emphasize student accomplishments.

Writing Activities

WRITING AND SOLVING MATH MYSTERIES

The teacher passes out to the students a very short mystery. The students are to solve this mystery by using the "Powers of Two Code".
Example:

The teacher was going to read a book to her students. She must wear her glasses to read the story. The teacher can not find her glasses. Where are her glasses?

The answer is in the code:

A = 1 ... Z = 26
Any letter can be found by using a power of two code.

1	2	4	8	16	
*					= A
*		*	*		1+4+8=13 = M

The answer to the mystery is:

1	2	4	8	16	
*	*	*	*		o
	*	*	*		n
			*		h
*		*			e
	*			*	r
			*		h
*		*			e
*					a
		*			d

Now ask the students to write their mystery and to provide the coded message to solve it. These can be put in the learning center for times when students have finished their assignments or they may trade mysteries and solve.
Extension: For older students, have the codes labels with powers of two, such as

2^0	2^1	2^2	2^3	2^4	(instead of
1	2	4	8	16)	

Sherron Frost
Katy, Texas

COPY CAT!

Allow students to make a design using Cuisenaire Rods, Attribute Blocks, or other manipulatives. After the design is complete, they should write down exactly how the design is laid out so that someone else could duplicate it using the instructions. Allow the students to exchange papers and try to copy each others' designs by reading the instructions.

Example:
"Lay down a square made of red rods. Put a white rod in the center. Put a light green rod at right angles to the red rod from the center of the red rod and radiating out. Do this on all four sides. Put a white rod at the end of each green rod."

RHYMING FACTS

Try this in your classroom:
Teacher says a fact like,
"Two plus two is four."
Student writes a rhyme for this, such as:

Two plus two is four
Who could ask for more?
or
Five times six is thirty
Don't get your bluejeans dirty!

Extension: To reinforce the students' addition, multiplication, subtraction, division facts (or any other mathematical drills) have them write the sequential facts with their rhymes in order.

ASK "MY DEAR AUNT SUZIE!"

In the classroom, set up a bulletin board area or a learning station and encourage the class to solve problems written to "My Dear Aunt Suzie" (Remember teachers, MDAS is the order of the mathematical operations.) Encourage students to create their own letters. Example:

> My Dear Aunt Suzie,
> I am eight years old and I am very much in love with an older woman who is one and a half times my age. Should I ask her to go steady or will she be too old for me when I reach eighteen?
> Signed,
> Lovelorn Larry

DAILY DOODLES

Have students keep a daily journal in which they use numbers, measurement, and other mathematical concepts.

The first few days the teacher may wish to get them started by providing a sample with blanks left to be filled in by the students.

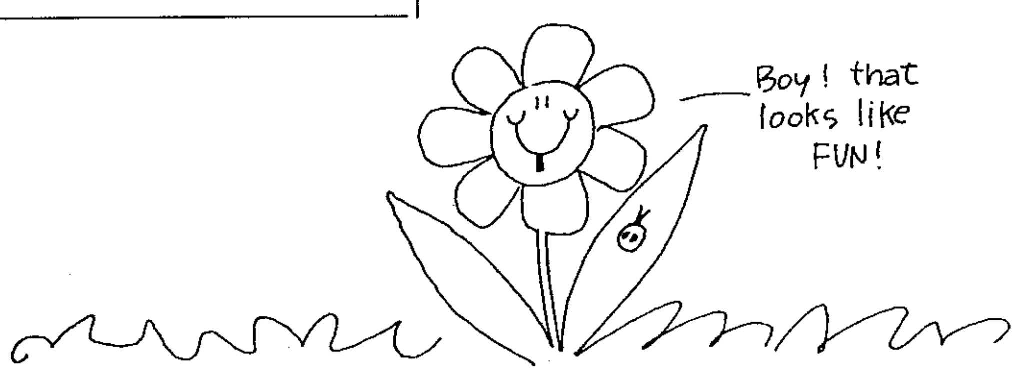
Example:
"Today we had ___ boys and ___ girls in the class. The temperature outside is ___. This is the ___ day of the month."

After the students are accustomed to writing daily, they can make up their own paragraphs.

For older students, the teacher may want to start a story and have the students finish it.

Example:
"I was flying to Zanziboo at 30,000 feet and the plane hit an air pocket and fell..." Have the students finish this story using as many numbers or mathematical problems as they can.

TRY IT!



It Works for Me

STORY PROBLEMS

Opportunities to use skills needed for writing and solving story problems do not surface frequently as a dyad. This reading-writing connection provides such an opportunity. As students listen to the story of a peer who has difficulty saving money, they may identify with him. Judith Viorst's talent for writing narratives in which fifth-grade children have real experiences keeps students interested as Alexander, Who Used to Be Rich Last Sunday is read aloud. Alexander's interaction with siblings, parents, and grandparents sets the stage for vocabulary extension, writing, addition, subtraction, multiplication, and division. (Essential Elements: 75.23 f1E, f2A-B, f3B 75.21 f1A, f4A11, f4B11, f4C1)

Focus: Students will write story problems involving money.

Materials: Alexander, Who Used to Be Rich Last Sunday. by Judith Viorst (Macmillan Publishing, New York)
Play money - bills and coins
Tokens
Writing tools, paper

Guided Practice: After the teacher reads the book, attention is given to the money in the story. Parts of the story are reread allowing the students to gather information for computation. How much money does Anthony have? How much money does Nicholas have? Students may use play money to solve problems. Model the writing of a story problem showing the grandparents' total gift to the three boys. Continue working with money as the students deduct the amounts Alexander spent and lost after he received $1.00. Various volunteers may put this information in sentence form on the chalkboard. Another student may add the question making the sentences a story problem.

Independent Practice: Students write a story problem including money. Reminded of the audience, their writings will be more interesting. A hidden answer key may be added. Individual conferencing for those needing help is appropriate now. Students may write as many story problems as they choose. One selection ready for trading with a classmate is due the next time this class meets.

Closure: In a grouping technique with four students per group, trade papers checking for sentences with information and a question. Make brief positive comments to the author. Trade papers again and repeat checks and comments. Praise and encourage all students for attempting to write story problems involving money. Students may take a token as a reminder to have the assignment ready to trade next time the class meets. (Tokens accompanying completed assignments may be redeemed for free time or a special privilege during next class.)

Cheryl Collins
Jimmy L. Elrod Elementary
San Antonio, TX

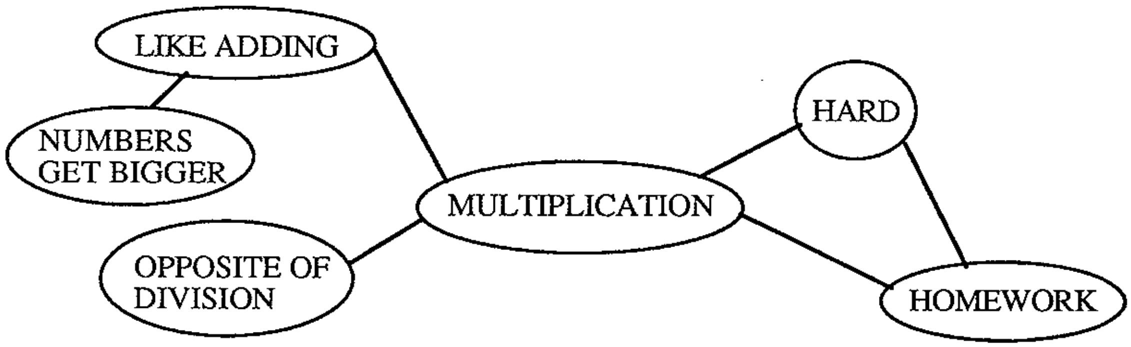
Mathematics - Write On!

Writing in the content areas is receiving increasing attention as an instructional tool for teachers and as a learning aid for students. Perhaps you have read an article on the topic yourself, used writing in other content areas besides mathematics, or recently taken a workshop on the writing process. But how does writing apply to the mathematics classroom? Why write in mathematics? What are some appropriate mathematics writing activities? This article will give you some reasons for using writing in mathematics and some practical ideas for incorporating writing in the mathematics classroom.

"Mathematics as Communication" is one of the standards in the National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards for School Mathematics*. Writing is one form of communication that offers many advantages. It allows the shy student, who may feel uncomfortable in an oral situation, the opportunity to express himself or herself in a private way. It contributes to skill in writing and reinforces the notion that writing is a lifetime skill needed in any field. Writing increases the learning of mathematics and promotes thinking skills. Through writing, students organize their thoughts and focus on important information. It makes abstract ideas more accessible to the student. Students can better understand and remember an idea by writing about it. Reviewing students' writing helps teachers to diagnose error patterns and/or mathematical misconceptions. Writing is a good activity at the beginning of the math lesson to help focus students' attention or at the end of a lesson to sum things up.

Having your students keep a journal is one way to incorporate writing in the mathematics classroom. You may ask older students to list goals for each six-week grading period. Students of any age could write about what they do best in mathematics. You may ask students to record interesting problems (including some not yet solved) and describe the strategies used or thought about. Journal writing can help students to work through their own explanations of new concepts, to define terms and symbols, and to comment on how they think they are doing. They can also use the journals to call for help or to ask for additional explanations. Focused writing, whereby students write to answer a question or center on a topic given by the teacher, is probably more productive than free writing. This calls for planning on the part of the teacher. Think about how journal writing could be used to help enhance your mathematics lessons. Work with other teachers to develop a list of appropriate journal writing ideas for your particular grade level.

Besides journal writing, there are other ways to use writing in the mathematics classroom. When your students do textbook or worksheet practice, have them end each assignment by taking one or more of the problems and writing a story problem that fits. Students can work together in cooperative learning groups to solve problems and write a report of their investigation. You may simply have students formulate questions for a specific set of facts or data. Written explanations of the processes used in solving a problem are often helpful to both student and teacher alike. Stream-of-consciousness writing while a problem is being solved provides a good record of a student's thinking. At all levels, including primary, writing or dictating to an adult can help students see connections, as illustrated in the network or web of all the words young students could think of regarding "multiplication."



from EQUALS, 1989

Mathematics provides interesting and challenging material about which to write, and writing adds a new spark of interest to the learning of mathematics. Try some of the ideas suggested in this article or perhaps now you can think of some others. Writing in mathematics helps to build students' understanding and gives teachers a rich information base from which they can make sound instructional decisions.

Becky McCoy, Elementary Mathematics Specialist, Texas Education Agency

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Fill out and mail to: Maggie Dement
4622 Pine
Bellaire, TX 77401

STEAM OFFICERS

Director:
Merry Brown
Brazosport ISD
Drawer Z, Freeport, TX 77541
(409) 265-6181

Program Chairman:
Anita Uphaus
Austin ISD
6100 Guadalupe, Austin, TX 78752
(512) 451-8411 ext. 204

TEA Consultant:
Becky McCoy
Elementary Mathematics Specialist
Texas Education Agency
1701 N. Congress Ave., Austin, TX 78701
(512) 463-9585

Editor:
Dorothy Ware
Galena Park ISD
P. O. Box 565, Galena Park, TX 77547
(713) 672-7491

Business and Community Representatives:
Cathy Rahlfs
Humble ISD
P. O. Box 2000, Humble, TX 77347
(713) 540-5000

Carolyn Felux
Marilyn Burns Education Associates
11707 Cooksey Rd., Adkins, TX 78101
(512) 649-2614

T.C.T.M.