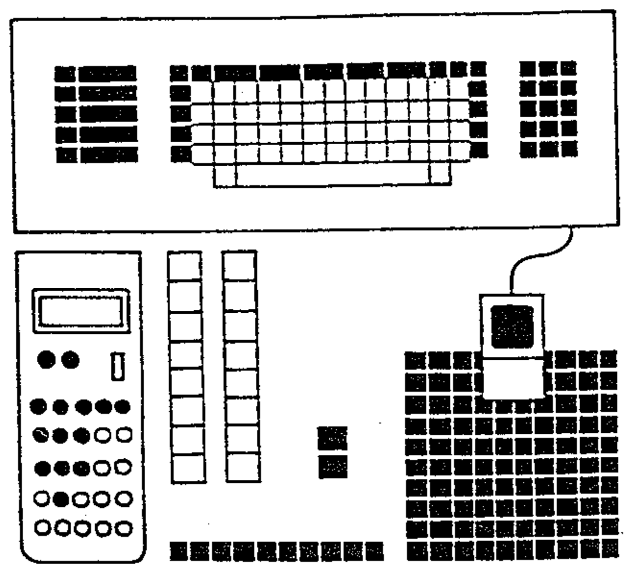
# TEXAS MATHEMATICS TEACHER



STUFF "Strategic Tactics Ultimately For Fun"

Metrication - Are We Still Involved?

Base Change by Synthetic Division

An Error Analysis of Two Similar Angle Trisection Methods

## MARCH 1989

# EDITORS



**TEXAS MATHEMATICS TEACHER**, the official journal of the Texas Council of Teachers of Mathematics, is published four times each year, in October, January, March, and May. All manuscripts and editorial correspondence should be addressed to George H. Willson, listed above.

Manuscripts should be neatly typewritten and double spaced, with wide margins, on 8 1/2 " by 11" paper. Authors should submit the signed original and four copies. Illustrations should be carefully prepared in black ink on separate sheets, the original without lettering and one copy with lettering added. As soon as possible after refereeing, authors will be notified of a publication decision. Originals of articles not accepted for publication will be returned to the authors. Two copies of the issue in which an author's manuscript appears will automatically be sent to the author.

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# TEXAS MATHEMATICS TEACHER

TEXAS COUNCIL OF TEACHERS OF MATHEMATICS

**Affiliated with the National Council of Teachers of Mathematics**

**March 1989**

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## President's Message

HELLOOOOOOO TEXAS TEACHEEEEEEEERS!!!!!!!

We have just a few more months to wait until the XXXVI annual meeting of CAMT in beautiful San Antonio, Texas. Yes, the riverwalk has been cleaned up, Sea World has it's mark, and now for all the ladies, a new mall located on the river. Leisure time, beautiful weather, wonderful sights, delicious food, a new mall and best of all, many new ideas to begin the school year. What else could one desire for a wonderful summer vacation and better yet, a tax write-off.

Now, down to business. It is election time again. It sure comes up often doesn't it? If you desire a real voice in TCTM and CAMT, it is necessary for you to send your suggestions for nominations to the nominating committee. Do you wish to be an officer and have worlds' of fun at the registration desk? Do you have a desire to fill in for missing speakers without a prepared topic; to fill in for missing presiders or find someone who will; to try to care for lost reservations; to find missing persons; to locate transportation and other wonderful fun activities? Well, just forward your name or the name of anyone in your district or state that would be an asset on the executive committee of TCTM. Send the name, school district, school, and home phone of potential candidates to Otto Bielss, 2609 Trinity Street, Irving, TX 75062.

Remember that CAMT is to be in San Antonio during August 2-4, 1989. TCTM will offer CAMT scholarships again this year. Scholarship information will be in the May journal. During CAMT, we need your help with several items, beginning with registration. We will try something new this year. We will allow those who pre-register to pick up their materials during the evening preceding the conference at the headquarters hotel, say 7:00 to 9:00 p.m. More in the May journal along with a volunteer form for other duties.

Plan to attend the TCTM breakfast on Wednesday morning, August 3, at 7:00 a.m. Again, more in the May journal.

A pre-conference meeting is being planned for CAMT much like the very successful one held last year in Houston. Watch for topics and time(s) in the May journal.

OTTO W. BIELSS, JR.
President, TCTM

## Perspectives:

We want to call special attention to the interesting articles in this issue of TMT.

Beverly Millican's "STUFF"article suggests a new twist in the use of games in the classroom. Presented is the idea of having students use their creativity in designing original games or adaptations of existing games. What a good way to give your students the opportunity to develop their imagination and thinking skills!

The article by L. Diane Miller is an excellent expose of where we are in implementing the Metric System in the U.S.A. The results of a survey done by Miller are presented; included is a copy of the instrument used. The article should suggest to you two questions: What is the status of metrication in your school? What are some actions you should be taking in your building?

Synthetic division related to numeration bases is the topic of Howard Lambert's article. This should prove useful to advanced students interested in using a different way of switching numeration bases.

The article by Harpold and Lamb is concerned with an age-old problem in geometry. The trisection of an angle seems to always challenge students and the authors provide useful knowledge in this area.

So Bon Appetite! If you have ideas other teachers would find useful, write an article and submit it to TMT for publication.

From your editors:

James Bezdek
George Willson

# STUFF

## ***Strategic Tactics Ultimately For Fun***

**Beverly Millican**

*Elementary Mathematics Coordinator*
*Plano Independent School District*

The last three issues of the TMT "STUFF"column have dealt with teaching a specific mathematical concept to students using games. Because students find games so motivating, designing the game itself can become the focus of an activity. One fifth grade teacher in Plano ISD, Sue Christian, has capitalized on her students' natural competitiveness, imagination, and creativity by assigning her class a "Design-a-Game" project.

In October, Sue explained the game project within the context of "real world" applications. Students analyzed math skills and the thinking required to play various commercial (or TV) games and were motivated to expand their classroom learning. Judging from the projects that her students presented in January, her objective was more than met!

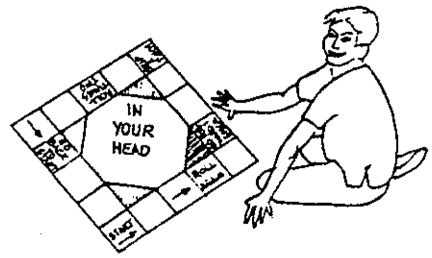
Sue's directions were simple in that each student was required to:

1) choose a concept utilizing previously mastered math skills,
2) design a game format (a variation of any commercial game or an original one),
3) create a game board or playing mat,
4) write clear directions, and,
5) make an oral presentation of the finished product.

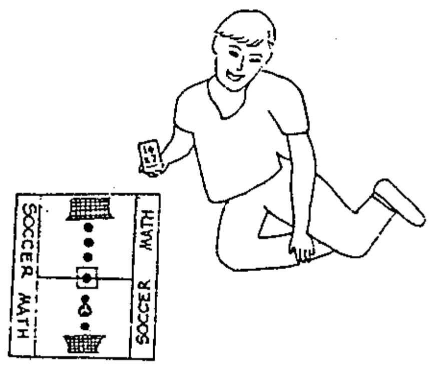
Among the games turned in were:

**"Mathentration"**- An adaptation of "Concentration,"the answer cards are placed face down on the playing board. Each player draws a question card, solves the problem, and attempts to locate the correct answer card on the "Mathentration" board.

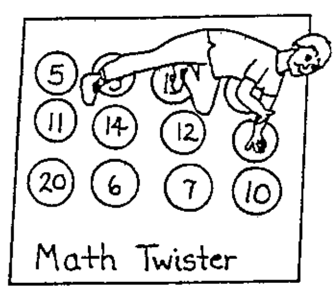
**"In Your Head"**- In this original game, each of the questions is a word problem. After tossing the die and moving ahead a number of spaces, each player has a time limit to work a problem and state the answer. If correct, he stays on the square, if not, he moves back one space.




**"Soccer Math"** - This "non-athletic" game uses flash cards. The first player to answer gets to move the soccer ball one space towards his team's goal.




**"Math Quizzard"** - An adaptation of "Trivial Pursuit," this game has categories that have been changed to number sentences, computation, measurement, Roman numerals, etc.




Other games turned in by students were adaptations of "Monopoly," "The Game of Life," "Twister," and even "Wheel of Fortune."

Each Friday afternoon, Sue allows her students to play the games in small groups. Although some games are more preferred than others by the class, Sue encourages her students to try them all and discuss what attributes of the games interest them the most.

Judging from her students' positive reactions to the "Design-a-Game" assignment, Sue has more than met her goal of "going beyond teaching computation skills and textbook pages" and expanding her students' awareness of real life mathematics skills.

Please Note . . . . . .

Apologies are in order for the late delivery of the January, 1989 issue of TMT. Printing/assembling/labeling problems plus the icy weather and flu season caused the three week delay.

The Editors

# METRICATION - ARE WE STILL INVOLVED?

**L. Diane Miller**

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*Baton Rouge, Louisiana*

President Gerald Ford signed the Metric Conversion Act into law in December, 1975. The purpose of the bill was to bring the metric system of measurement into common usage in the United States. In the 1976 Yearbook of NCTM, Sawada and Sigurdson proclaimed that by 1985 the United States should be completely converted to using the seven basic units of the Systeme International des Unites (SI). At the beginning of 1976, educators hypothesized that school children would lead the nation through the whole process of metrication (the period of change-over from customary units of measure to metric units of measure). Many believed that just as the "new mathematics" movement caused a complete revision of the curriculum, so would the movement toward the use of metric measures (Sawada & Sigurdson, 1976).

Some people, teachers included, view metrication as a fad that will pass just as they have witnessed the abandonment of "new math." While the conversion of the United States to common usage of metrics has progressed slowly, the conversion will almost certainly occur. Why? Because the metric system of measurement has several distinct advantages over our present system of measurement including: (1) metric measurements are simple and logical; (2) within metrics, larger and smaller units are established by a logical system of prefixes based on tens; (3) it is easier to compute and convert measurement in metrics than it is in the customary system; (4) being a decimal system, the use of decimal fractions (0.5, 2.25) eliminates a problem area for children who have difficulty understanding and computing common fractions (1/12, 1/8); and (5) metrics is the measurement system of international trade (Hovey & Hovey, 1983). Competition in the international market is adversely affected by the dual productionlines in the United States - one in the customary system and one in metrics. These adverse effects in international trade also negatively alter America's domestic economy, which can and does affect consumers on a daily basis. For a more complete overview, Hovey and Hovey (1983) present an excellent historical perspective of SI in addition to outlining the current status of metrication.

One can argue on an intellectual basis why the United States should completely convert to a metric system of measurement; but what is being done educationally to realize this goal of complete conversion? In an attempt to answer this question, a survey of Algebra I and Consumer Mathematics students was conducted at two high schools in Louisiana. One school is public school with 99% of its student body being black. A high percentage of the student body participates in the school's free lunch program. The other school is a university laboratory school with 86% of its student body being white. A high percentage of the student body comes from families with above average incomes. The curriculum of the laboratory school is considered college preparatory.

The reasons for selecting these two schools were twofold. The author wanted to survey students coming from two different backgrounds and physical access to these two schools was convenient. The survey instrument was administered to two Algebra I classes (n = 32) and three Consumer Mathematics classes (n = 57) at the public school and one Algebra I class (n = 30) at the laboratory school.

The survey instrument and the number of students responding to each item are shown in Table 1. The first eight items of the instrument are from A Metric Handbook for Teachers (Shumway, Sacks, & Higgins, p. 40). Nine of the last twelve items (11, 12, 13, 15, 16, 17, 18, 19, 20) were included in the survey to present the student with additional settings in which metric measures might be encountered on a daily basis. Items 9 and 10 were included to ascertain the students' understanding of just "how much" is a gram and "how high" is a meter. Item 14 was included to determine if students knew the relationship between metric volume (a liter) and metric mass (1000 grams). Items 9-20 were collected from various literature reviews over a period of years.

Table 1.

Think Metric -- Live Metric

Directions: Circle the answer which you think is the most reasonable estimate.

The correct responses have been identified by an asterisk, *.
The number of students responding to each item is shown to the right.

| | Con Math Pub Sch | Alg I Pub Sch | Alg I Lab Sch |
|---|---|---|---|
| 1. The height of a man playing center on a typical high school basketball team is approximately. | | | |
| a) 6 m | 30 | 15 | 4 |
| b) 240 m | 7 | 8 | 1 |
| * c) 2 m | 5 | 1 | 19 |
| d) 78 m | 15 | 8 | 6 |
| 2. My car was a little low on oil so the gas station attendant recommended that I add a can of oil containing | | | |
| a) 2 mL | 10 | 6 | 1 |
| * b) 1 L | 20 | 9 | 20 |
| c) 10 mL | 18 | 11 | 8 |
| d) 20 L | 9 | 6 | 1 |
| 3. The diameter of a coffee cup is about | | | |
| a) 1 cm | 14 | 6 | 2 |
| * b) 8 cm | 30 | 13 | 17 |
| c) 20 cm | 7 | 9 | 8 |
| d) 50 cm | 5 | 3 | 3 |
| no response circled | 1 | 1 | 0 |

| | | | |
|---|---|---|---|
| 4. In order to bake a pizza, one should set the oven temperature at about | | | |
| a) 100° C | 4 | 2 | 4 |
| b) 400° C | 37 | 20 | 15 |
| * c) 220° C | 15 | 9 | 7 |
| d) 600° C | 0 | 1 | 4 |
| no response circled | 1 | 0 | 0 |
| 5. A good weight for a college-age girl of average height would be about | | | |
| a) 138 g | 30 | 16 | 4 |
| b) 40 g | 5 | 5 | 3 |
| c) 150 kg | 16 | 10 | 6 |
| * d) 55 kg | 5 | 1 | 17 |
| no response circled | 1 | 0 | 0 |
| 6. The capacity of a classroom aquarium is usually about | | | |
| a) 10 L | 5 | 1 | 10 |
| b) 200 mL | 19 | 11 | 6 |
| * c) 40 L | 25 | 10 | 12 |
| d) 25 mL | 8 | 10 | 2 |
| 7. The length of a car is approximately | | | |
| * a) 5 m | 5 | 5 | 20 |
| b) 15 mm | 14 | 7 | 0 |
| c) 26 m | 36 | 20 | 10 |
| c) 3 cm | 2 | 0 | 0 |

| | | | |
|---|---|---|---|
| 8. A good temperature to set your home thermostat at for comfortable living would be | | | |
| a) 90° C | 3 | 1 | 4 |
| b) 32° C | 13 | 3 | 14 |
| c) 70° C | 36 | 24 | 8 |
| * d) 20° C | 5 | 4 | 4 |
| 9. A gram is about the mass of | | | |
| a) an apple | 4 | 4 | 6 |
| b) a grain of sugar | 25 | 15 | 5 |
| c) a graham cracker | 7 | 2 | 2 |
| * d) two paper clips | 21 | 11 | 17 |
| 10. A meter is about the height of | | | |
| a) a door | 12 | 13 | 4 |
| * b) a door knob | 18 | 13 | 14 |
| c) a chair seat | 17 | 2 | 8 |
| d) an adult person | 10 | 4 | 4 |
| 11. A measuring cup would hold about | | | |
| a) 2.5 mL | 20 | 9 | 9 |
| b) 25 mL | 20 | 16 | 11 |
| * d) 250 mL | 16 | 7 | 10 |
| no response circled | 1 | 0 | 0 |
| 12. A newborn baby has a mass of about | | | |
| * a) 3 kg | 12 | 11 | 17 |
| b) 30 kg | 38 | 12 | 11 |
| c) 300 kg | 6 | 9 | 2 |
| no response circled | 1 | 0 | 0 |

| | | | |
|---|---|---|---|
| 13. Normal body temperature is about | | | |
| a) 25° C | 7 | 4 | 1 |
| * b) 37° C | 17 | 12 | 14 |
| c) 45° C | 33 | 16 | 14 |
| no response circled | 0 | 0 | 1 |
| 14. A liter of water has a mass of about | | | |
| a) 10 g | 5 | 6 | 2 |
| b) 100 g | 31 | 13 | 14 |
| * c) 1000 g | 14 | 7 | 13 |
| d) 1000 kg | 6 | 6 | 1 |
| no response circled | 1 | 0 | 0 |
| 15. A soda straw is about how long | | | |
| a) 2 mm | 23 | 12 | 2 |
| b) 20 mm | 26 | 14 | 18 |
| * c) 200 mm | 8 | 6 | 9 |
| no response circled | 0 | 0 | 1 |
| 16. A pro-football player has a mass of | | | |
| a) 45 kg | 3 | 1 | 2 |
| * b) 90 kg | 11 | 10 | 10 |
| c) 225 kg | 32 | 17 | 10 |
| d) 180 kg | 11 | 4 | 8 |
| 17. A dime is about how thick | | | |
| a) 0.1 mm | 27 | 16 | 11 |
| * b) 1 mm | 23 | 10 | 17 |
| c) 10 mm | 7 | 6 | 1 |
| no response circled | 0 | 0 | 1 |

| | | | |
|---|---|---|---|
| 18. A teaspoon holds about | | | |
| a) 0.5 mL | 22 | 10 | 14 |
| * b) 5 mL | 30 | 15 | 15 |
| c) 50 mL | 2 | 2 | 1 |
| d) 1 L | 3 | 5 | 0 |
| 19. A woman's waist measurement is about | | | |
| a) 64 mm | 14 | 2 | 13 |
| * b) 64 cm | 29 | 16 | 15 |
| c) 64 m | 14 | 14 | 2 |
| 20. An automobile gasoline tank would hold about how much gasoline | | | |
| a) 80 ml | 7 | 4 | 4 |
| * b) 80 liters | 31 | 12 | 17 |
| c) 80 kiloliters | 18 | 8 | 2 |
| d) 80 centiliters | 1 | 8 | 7 |

The survey instruments were scored as number correct. The mean scores are shown in Table 2. There was no significant difference ($p < .01$) between the means of the Algebra I and Consumer Math students at the public school.

Table 2

Survey Mean Scores - Number Correct out of 20

| | Algebra I | Consumer Math |
|---|---|---|
| Public School | 5.63* (n = 32) | 5.93 (n = 57) |
| Laboratory School | 9.60* (n = 30) | --- |

*significant difference at $P < .01$

While the Algebra I laboratory school mean is significantly higher ($p < .01$) than the Algebra I public school mean, scoring less than 50 percent correct on this survey instrument is still cause for alarm. A careful examination of each item will reveal that in almost every case, there is only one reasonable estimate. For example, look at the first item. The classes surveyed are predominantly ninth graders with a few upper classmen enrolled. Almost all of these students are familiar with the sport of basketball; either through personal participation or as a spectator. Hopefully, each of these students have seen a meter stick and know its relationship to a yard stick. Yet, over 40 percent of this group chose six meters as the correct response to this item. This answer is far from reasonable. People who look at this item and the response selected most frequently may say, "They (the students) were thinking about feet." On the contrary, one alarming aspect about the results of this survey is the apparent lack of thought that is revealed by the answers given. Looking at item two, there is only one reasonable answer. Milliliter is a measure comparable to a teaspoon. Liter is a measure familiar to many people through the sale of milk, soft drinks and other beverages. Yet nearly 60 percent of the students made the wrong choice for this item. One might excuse the students' incorrect response because of their lack of knowledge about cars and the measure of oil that cars use. However, when considering the age of these students, one begins to question not only the exposure to metric measures being provided in schools but also the exposure and experiences students get outside of school.

One could continue with an item analysis of this survey instrument and hypothesize why the students' performance was so poor. The incorrect answers of some items may be attributed to the students thinking about customary measures rather than metric units. Only four of the 20 items resulted in 50% or more of the students selecting the correct response. Follow-up interviews with students would have been helpful in understanding why the students selected the answers given. The author would suggest to anyone conducting a similar survey of their students' knowledge about the metric system to include time for follow-up interviews. The value of knowing <u>why</u> students have missed items far surpasses the value of just knowing that they have missed them.

The author recognizes that a few of the items on the survey instrument are outdated. For example, having recently watched Super Bowl XXII, the correct response to item 16 should probably be more than 225 kg. With the recent influx of compact cars into our society, the length of a car in item seven may need revising. However, if one considers the three foils in each of these two items, there really is only one answer that is reasonable. The overall results of this informal survey indicate that this group of students have no "metric sense." They are probably not unique to other high school students in the nation. What should this suggest to teachers? We have a lot of work to do before the results of this or similar surveys will be reversed. It is very easy to pass the responsibility of teaching the metric system to someone else. Secondary teachers may feel that metrication is the responsibility of middle school teachers. Middle school teachers are presently working with a very crowded curriculum and may feel that the metric system should be taught in the elementary grades. Mathematics teachers may place the responsibility on science teachers; however, science teachers generally question why mathematics teachers do not teach the metric system in their math classes. As all teachers know, "passing the buck" does not solve the problem. It is a task that all teachers should undertake.

The National Council of Teachers of Mathematics (NCTM) took an official position on the metric system and the teaching of measurement in 1983. The following statement, developed by the Instructional Issues Advisory Committee and adopted by the Board of Directors, was published in the September, 1983 issue of the NCTM newsletter.

> The NCTM strongly supports the adoption and use of the metric system of measurement, and its teaching at all educational levels
>
> Powerful forces dictate that national economic health is dependent on world trade. The metric system is the required standard of that international commerce. Further, the metric system strengthens and is compatible with the teaching of the decimal system for numbers and currency.
>
> Measurement instruction is more than teaching a specific system of units. Thus, measurement experiences should evolve from nonstandard units through the general concept of an arbitrary unit to a standard metric unit. The specific developmental teaching of

the metric system should be accompanied by other general ideas of measurement such as congruence and covering.

Metric education is critical for communication in our global society.

NCTM reiterated its support for metrication in 1986 with the following position statement.

Measurement is a common daily activity performed in all nations, and a knowledge of its concepts and skills is an important topic for all sectors of society. On an international level, the increased demand for a common language of measurement has been eased by the adoption of the metric system. The teaching of this system of measurement is not the sole responsibility of mathematics teachers, however; all teachers and administrators share the responsibility to prepare students for a world where the metric system of measurement prevails. Yet, many in education, including mathematics education, continue to ignore the teaching and learning of the metric system.

As a part of the mathematics program, metric education provides models for activities involving numeration, decimal concepts, and estimation. It serves as a link that joins measurement concepts and skills to other major topics in the school mathematics curriculum.

Therefore, the National Council of Teachers of Mathematics supports the use of the metric system as an integral part of the mathematics curriculum at all levels of education and recommends that all educators assume responsibility for providing leadership and direction in metric education.

In support of this position, all teachers have a responsibility to adequately prepare students for the metric world in which they live. Following are a few suggestions that may help to do so.

1. Make metrics meaningful! Introduce metric measures through everyday experiences. Illustrate to students how metric measures

already permeate their life. A few examples include: soft drinks and other beverages are available in liter quantities; the temperature flashed on signs by community businesses is often given in Celsius degrees; and, many road signs now register distances between points in kilometers as well as miles.

2. Get parents and/or other family members involved in activities related to the recognition of the use of metrics in everyday life. Give students a check-list of items that can be found in the home that reflect metric measures. For example: a liter of motor oil, a bathroom scales showing kilograms and pounds, a speedometer registering kilometers and miles, kitchen thermometers listing both degrees Celsius and degrees Fahrenheit, canned goods reflecting measures in ounces and grams, tools that denote their size in centimeters, recipes that list metric and standard measures, etc. Challenge each student to list an item in their home that represents a metric measure that no one else may have. Students of all ages generally enjoy this type of competition.

3. Provide students with opportunities to measure the length, mass, and volume of a variety of objects including their own body measurements. These measurements will differ between individuals, but the idea is to be able to use body measurements to make estimates of other measures. Being able to make reasonable comparisons can help people make decisions in everyday life. Allow students the opportunity to decide if given objects are bigger, smaller, longer, shorter, heavier, lighter or the same as others. Even close approximations of angle measures can be accomplished with body measurements. Butting the second knuckles of the two middle fingers together and making a triangle with the two forefingers roughly approximates an equilateral triangle. These 60-degree angles can be surprisingly accurate. Finally, by making the old V for victory sign with the forefinger and middle finger, an angle of approximately 45-degrees is formed (The Milwaukee Journal, 1983).

4. Encourage students to develop a "metric sense." A feel for the relative comparison between "meter" and "yard" or "kilometer" and "mile" will be helpful. High school students should know that six meters is an unreasonable height for a basketball player. Create other examples in which students must choose the appropriate label for a given measure

or supply the measure if given the label. Two examples are: A newborn baby has a mass of about 3 __. Students can select either grams or kilograms to complete the statement; or, rephrase the statement: A newborn baby has a mass of about __ kg. Students can select either 3 or 30 to fill-in the blank. A few statements of this type can be put on a transparency for students to complete while the teacher checks roll at the beginning of class, as students prepare for lunch or recess, etc. The suggestion of frequent, repeated practice leads to the next item.

5. A person's metric education cannot be completed overnight or through any short term program. Activities with metric measures should be included throughout the curriculum at all grade levels. Furthermore, the instructional responsibility should not lie solely with the mathematics and/or science teachers. The history of the metric system is a good topic for discussion in history and/or social studies classes. The advantages of adopting SI measures in the United States can be covered in an economics or free enterprise class. Metric measures should be used in home economics, industrial arts classes, and any vocational education course where measurements are required.

6. Finally, before students can be expected to lead our nation through metrication, many teachers must become more familiar with the system. The ultimate goal for all of us is to "think metric." Memorizing conversion factors and algorithms for calculating conversions is not the way to proceed. Through the activities suggested for the students, teachers, too, can learn to think metric.

We must all assume the attitude represented by a well know phrase: "The buck stops here." The metric system of measurement is more logically constructed, more rationally ordered, and more convenient to use than our customary system of measurement (Hovey & Hovey, 1983). As teachers, we must all accept the responsibility of helping our students understand and use this better system of measurement. As citizens, we must all be involved in metrication.

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# BASE CHANGE BY SYNTHETIC DIVISION


**Howard B. Lambert**

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*Commerce, Texas*


We are sometimes asked to convert a number, written in one base to another base. One common way is to change the given numeration to base 10 and then change the base 10 numeration to the desired base. This may very well be the optimal way with respect to time and accuracy, but is wasteful in terms of the number of steps required. We show here a method, using synthetic division which reduces the number of arithmetic steps.

It is assumed that the reader is familiar with synthetic division, but a quick refresher is in order. Consider the different groupings of the following polynomial:

$$P(X) = 2X^3 + 3X^2 + 4X + 5 \quad (1)$$
$$= (2X^2 + 3X + 4)X + 5$$
$$= ((2X + 3)X + 4)X + 5 \quad (2)$$

Now suppose P(X) is to be evaluated at, say $X = 6$, then using (1)

$$P(X) = 2(6)^3 + 3(6)^2 + 4(6) + 5$$
$$= 2(216) + 3(36) + 4(6) + 5 = 569$$

But equation (2) suggests the technique known as synthetic division.

| | | | | |
|---|---|---|---|---|
| 6 | 2 | 3 | 4 | 5 |
| | | 12 | 90 | 564 |
| | 2 | 15 | 94 | 569 |

The 2, 3, 4, and 5 are the coefficients of the polynomial and the pattern is: bring down the first coefficient 2, multiply by 6, add to the next coefficient, multiply by 6, add to the next coefficient, etc. Evaluating a polynomial in the form of equation (2) is Horner's Method and is more efficient as far as the number of arithmetic steps requires.

The base 10 numeration 2345 actually stands for

$$2(10)^3 + 3(10)^2 + 4(10) + 5.$$

Since this is the form of a polynomial, we define

$$P(X) = 2X^3 + 3X^2 + 4X + 5$$

to be the polynomial associated with the numeration 2345.

$$\text{Let } P(X) = a_nX^n + \ldots + a_1X + a_0$$

be the polynomial associated with the number whose digits are

$$a_n \ldots a_1a_0 \text{ in base A,}$$

usually written $a_n \ldots a_1a_0$ (A). Clearly

$$P(A) = a_n \ldots a_1a_0 \text{ (A).}$$

If the a's represent fixed numbers, the value of P(X) depends only on the numerical value of X and not on the base in which the arithmetic is performed. The value of P(X) would, however, be expressed in the base of the arithmetic. The evaluation of P(X) for a given X suggests the use of synthetic division. The following algorithm will convert from base A to base B.

Algorithm: Given a base A numeration for a number, write in synthetic division form, the polynomial associated with the number, but express all coefficients in base B. Evaluate at A, where A is expressed in base B, doing the arithmetic in base B.

The following three examples will illustrate the algorithm.

Example 1. Change 2102(3) to base 10.

Base 3 digits are base 10 digits so we write:

| | | | | |
|---|---|---|---|---|
| 3 | 2 | 1 | 0 | 2 |
| | | 6 | 21 | 63 |
| | 2 | 7 | 21 | 65 |

Arithmetic in base 10.

Thus 2102(3) = 65(10).

Example 2. Change 2102(3) to base 4.

Base 3 digits are base 4 digits so we write:

| | | | | |
|---|---|---|---|---|
| 3 | 2 | 1 | 0 | 2 |
| | | 12 | 111 | 333 |
| | 2 | 13 | 111 | 1001 |

Arithmetic is in base 4.

Thus 2102(3) = 1001(4)

Example 3. Change 2102(3) to base 2. The digits must change to base 2, i.e., 2 expressed as 10(2) and 3 as 11(2).

| | | | | |
|---|---|---|---|---|
| 11 | 10 | 1 | 0 | 10 |
| | | 110 | 111 | 10101 |
| | | | 111 | 10101 |
| | 10 | 111 | 10101 | 1000001 |

Arithmetic in base 2

Thus 2102(3) = 1000001(2).

Not only is the above method of changing bases more efficient as far as the number of arithmetic steps required, but it gives the student a chance to do arithmetic in bases other than 10.

# AN ERROR ANALYSIS OF TWO SIMILAR ANGLE TRISECTION METHODS

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When students in a geometry class study constructions that are possible with a straight edge and compass, they usually notice the following situation: it is easy to bisect a segment, it is easy to bisect an angle and it is not very difficult to trisect a segment. Therefore, it seems plausible it should be possible to trisect an angle. The following method is often tried:

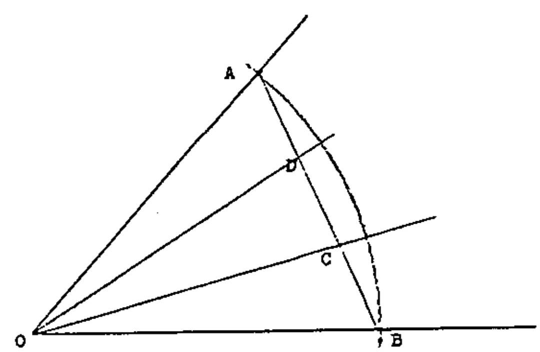


Figure 1

Given angle AOB in Figure 1, we can assume that AO = OB. Connect A and B and trisect segment AB at C and D. Draw OC and OD. It appears that angle BOC, COD and DOA are each one-third of angle AOB.

We must now determine if the method works for all angles, some angles or no angles. In Figure 2, let OA = 1, t = the measure of angle AOB, O have coordinates (0,0), and B have coordinates (1,0), so A has coordinates (cos t, sin t).

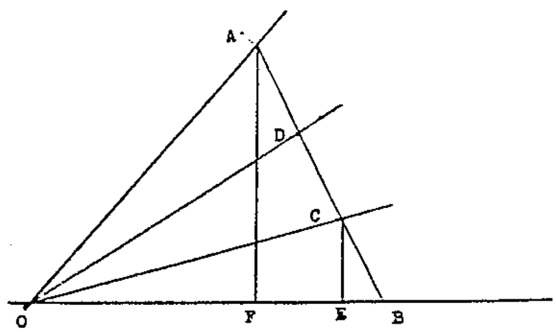


Figure 2

We need the coordinates of C. Drop a perpendicular from A to OB at F and drop a perpendicular from C to OB at E. Notice that OF = cos t. Since FE/2 = FB/3, then FE = (2/3)(1 - cos t). Thus OE = (2 + cos t)/3. Also, CE = (1/3) sin t, so tan COE = (1/3)(sin t)/(1/3)(2 + cos t) = sin t/(2 + cos t). Therefore,

$$\text{angle } COE = \arctan(\sin t/(2 + \cos t)). \qquad (1)$$

If the method works for an angle of measure t, then COE should be t/3.

The following BASIC computer program that was written for a TRS 80 Model III checks the measure of angles between 0 and 180 degrees that are multiples of 5 degrees to find any error made by the method:

# PROGRAM LISTING FOR THE HARPOLD TRISECTION METHOD

```
5 LPRINT "                    ERROR ANALYSIS OF THE HARPOLD ANGLE TRISECTION METHO
D"
6 LPRINT:LPRINT
10 CLS : LPRINT "    ANGLE     1/3 ANGLE     CONSTRUCTED 1/3          ERROR
     % ERROR"
20 LPRINT : LPRINT "                                                 DEG  MIN
 SEC"
30 LPRINT
40 U$ = "    ###       ##.###          ###.###          ##   ##   ##      #
#.######"
50 FOR I = 0 TO 180 STEP 5
60 Z = I*3.141592/180 : REM - CONVERT DEGREES TO RADIANS
70 X = I/3 : REM - ACTUAL THIRD
80 Y = ATN(SIN(Z)/(2 + COS(Z)))
90 W = Y*180/3.141592 : E = ABS(W - X) : REM - CONVERT RADIANS TO DEGREES AND
 COMPUTE ERROR
100 W1 = INT(E) : W2 = INT(60*(E-W1)) : W3 = INT(60*(60*(E-W1)-W2)) : REM - S
EPARATE E INTO DEGREES, MINUTES AND SECONDS
105 IF I = 0 THEN PE = 0 : GOTO 120
110 PE = 100*E/I : REM - COMPUTE PRECENTAGE OF ERROR
120 LPRINT USING U$;I,X,W,W1,W2,W3,PE
130 NEXT I
```

The output of the program is the following table:

ERROR ANALYSIS OF THE TWO ANGLE TRISECTION METHODS

| Angle | 1/3 Angle | Constructed 1/3 Angle COB Fig.1 Angle COG Fig.3 | Error Deg | Min | Sec | % Error Based on original angle. (First Column) |
|---|---|---|---|---|---|---|
| 0 | 0.000 | 0.000 | 0 | 0 | 0 | 0.000000 |
| 5 | 1.667 | 1.666 | 0 | 0 | 1 | 0.009422 |
| 10 | 3.333 | 3.330 | 0 | 0 | 13 | 0.037715 |
| 15 | 5.000 | 4.987 | 0 | 0 | 45 | 0.085100 |
| 20 | 6.667 | 6.636 | 0 | 1 | 49 | 0.151968 |
| 25 | 8.333 | 8.274 | 0 | 3 | 34 | 0.238834 |
| 30 | 10.000 | 9.896 | 0 | 6 | 14 | 0.346365 |
| 35 | 11.667 | 11.500 | 0 | 9 | 59 | 0.475439 |
| 40 | 13.333 | 13.082 | 0 | 15 | 3 | 0.627115 |
| 45 | 15.000 | 14.639 | 0 | 21 | 40 | 0.802655 |
| 50 | 16.667 | 16.165 | 0 | 30 | 6 | 1.003570 |
| 55 | 18.333 | 17.656 | 0 | 40 | 38 | 1.231660 |
| 60 | 20.000 | 19.107 | 0 | 53 | 36 | 1.488990 |
| 65 | 21.667 | 20.511 | 1 | 9 | 20 | 1.778020 |
| 70 | 23.333 | 21.862 | 1 | 28 | 16 | 2.101590 |
| 75 | 25.000 | 23.153 | 1 | 50 | 50 | 2.463030 |
| 80 | 26.667 | 24.374 | 2 | 17 | 34 | 2.866210 |
| 85 | 28.333 | 25.515 | 2 | 49 | 5 | 3.315640 |
| 90 | 30.000 | 26.565 | 3 | 26 | 5 | 3.816610 |
| 95 | 31.667 | 27.510 | 4 | 9 | 23 | 4.375270 |
| 100 | 33.333 | 28.334 | 4 | 59 | 55 | 4.998840 |
| 105 | 35.000 | 29.019 | 5 | 58 | 49 | 5.695750 |
| 110 | 36.667 | 29.543 | 7 | 7 | 24 | 6.475830 |
| 115 | 38.333 | 29.880 | 8 | 27 | 11 | 7.350580 |
| 120 | 40.000 | 30.000 | 9 | 59 | 59 | 8.333330 |
| 125 | 41.667 | 29.867 | 11 | 47 | 57 | 9.439360 |
| 130 | 43.333 | 29.441 | 13 | 53 | 30 | 10.686000 |
| 135 | 45.000 | 28.675 | 16 | 19 | 29 | 12.092500 |
| 140 | 46.667 | 27.516 | 19 | 9 | 3 | 13.679200 |
| 145 | 48.333 | 25.907 | 22 | 25 | 33 | 15.466200 |
| 150 | 50.000 | 23.794 | 26 | 12 | 21 | 17.470700 |
| 155 | 51.667 | 21.127 | 30 | 32 | 21 | 19.702800 |
| 160 | 53.333 | 17.878 | 35 | 27 | 19 | 22.159600 |
| 165 | 55.000 | 14.052 | 40 | 56 | 52 | 24.817000 |
| 170 | 56.667 | 9.707 | 46 | 57 | 36 | 27.623600 |
| 175 | 58.333 | 4.962 | 53 | 22 | 15 | 30.497700 |
| 180 | 60.000 | 0.000 | 59 | 59 | 59 | 33.333300 |

Note that the method trisects angles up to 60 degrees with an error of less than one degree for angle BOC. However, as larger angles are considered, the error grows to 10 degrees by the time we reach 120 degrees. Then disaster strikes. For angles between 120 degrees and 180 degrees, the method produces angles that are less than 30 degrees in descending order so that by the time we reach 180 degrees, the error is 60 degrees. We will now show that this apparent maximum of 30 degrees at 120 degrees is in fact the largest angle that this method can produce. Taking the derivative of

$$y = \tan^{-1}(\sin x/(2 + \cos x))$$

we have

$$y' = (1/(1+(\sin x/(2 + \cos x))^2)((2 + \cos x)(\cos x) -$$

$$(\sin x)(-\sin x))/(2 + \cos x)^2.$$

Simplifying and setting $y' = 0$, we get

$$(2 \cos x + 1)/(5 + 4 \cos x) = 0.$$

Since $5 + 4 \cos x$ is never zero, we can set

$$2 \cos x + 1 = 0$$

so $\cos x = -(1/2)$ and thus $x = 120°$

as we expected from the computer output.

In order to find the angles that the method actually trisects, we return to equation (1) and assume angle COE is t/3. Thus

$$\tan t/3 = \sin t/(2 + \cos t).$$

To eliminate fractions, let $x = t/3$ so $3x = t$. Now we have:

$$\tan x = \sin(3x)/(2 + \cos 3x)).$$

Substituting the identities for sin 3x and cos 3x, we get:

$$\tan x = (3 \sin x - 4 \sin^3 x)/(2 + 4 \cos^3 x - 3 \cos x),$$

so $$\sin x/\cos x = (3 \sin x - 4 \sin^3 x)/(2 + 4 \cos^3 x - 3 \cos x).$$

If we assume sin x is not zero, we have:

$$1/\cos x = (3 - 4 \sin^2 x)/(2 + 4 \cos^3 x - 3 \cos x).$$

Cross multiplying and substituting $1 - \cos^2 x$ for $\sin^2 x$, we get

$$2 + 4 \cos^3 x - 3 \cos x = 3 \cos x - 4 \cos x + 4 \cos^3 x$$

so $$2 - 3 \cos x = - \cos x.$$

Thus $\cos x = 1$ so $x = 0°$.

Therefore, the only angle that is actually trisected by this method is the zero-degree angle.

Another method to approximately trisect an angle with straight edge and compass discovered by Lynn Harpold[1] has a similar analysis. The construction is as follows:

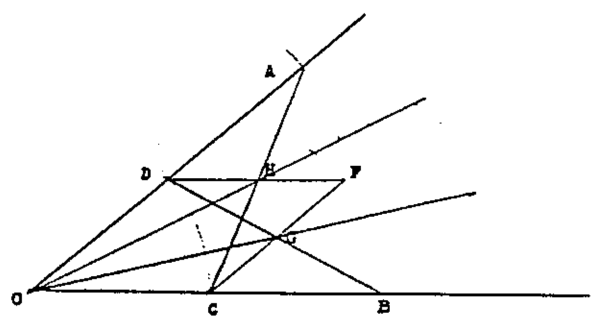


Figure 3

Given angle AOB with measure t, we can assume AO = 2 and OB = 2. Let the midpoints of AO and OB be D and C respectively. Construct the bisector of angle AOB using the radius 1 for all arcs so that OD = OC = DF = CF which means that quadrilateral OCFD is a rhombus. Now draw AC and BD. Let AC meet DF at H and let BD meet CF at G. It can be claimed that angles BOG, GOH and HOA are each one-third of angle AOB.

Let the various points in Figure 3 have the following coordinates: O has coordinates (0,0); C has coordinates (1,0); B has coordinates (2,0) and

D has coordinates (cos t, sin t). Since DF is parallel and congruent to CB, quadrilateral CBFD is a parallelogram. Thus the diagonals BD and CF bisect each other. The coordinates of F are (cos t + 1, sin t), so the coordinates of G are ((cos t + 2)/2, (sin t)/2). Thus

$$\tan GOB = ((\sin t)/2)/((\cos t + 2)/2)$$

so

$$\tan GOB = (\sin t)/(2 + \cos t).$$

Thus

$$\text{angle } GOB = \arctan((\sin t)/(2 + \cos t))$$

which is identical to equation (1) obtained for the first method.

Therefore, the error analysis of the Harpold Method is the same as for the previous method. Thus the Harpold Method has an error of less than one degree for angles less than or equal to 60 degrees and works exactly for only the zero-degree angle.

One may wonder if it is just a coincidence that these two methods produce the same size angle for an approximate third of a given angle. It turns out that the Harpold Method actually trisects the segment used in the first method. This can be proved as follows using Figure 4.

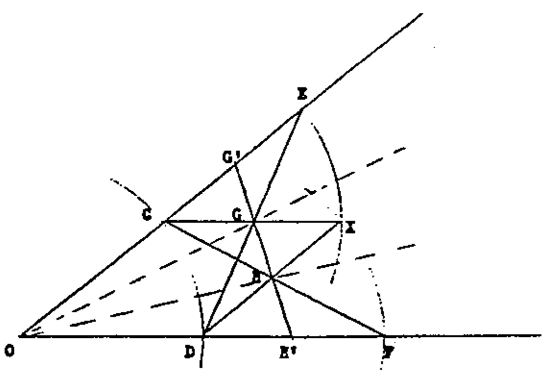


Figure 4

Figure 4 is a copy of Figure 3 with different letters for the points. Let the segment joining G and H meet the sides of the given angle at G' and H' as shown. Note that ODXC is a rhombus since all sides are equal. Therefore, OC is parallel to DX, so CE is parallel to DX. Since DX and

CE are also equal, CDXE is a parallelogram. In a similar manner, we can show CDFX is also a parallelogram. Diagonals DE and CX bisect each other at G as do diagonals CF and DX at H. Triangles CGG' and XGH are congruent by ASA, so GG' - GH. Likewise, triangles CHG and FHH' are congruent by ASA, so GH = HH', Thus, G and H trisect G'H'. This result can be generalized to say that the rays OG and OH trisect any segment between points on the sides of the given angle that is perpendicular to the bisector of the given angle.

It is also interesting to note that equation (1) is similar to an equation obtained from the analysis of a Trisection Method discovered by d'Ocagne.[4] By this method, the equation for one-third of the given angle with measure t is:

$$\tan t/3 = (2 \sin (t/2))/(1 + 2 \cos (t/2)).^2$$

This method is more accurate than the Harpold Method. It has errors less than one degree for angles up to 120°.[2]

There are several other methods that approximately trisect a given angle.[4] Some of them actually work for specific angles, but none of them will work for all angles, because algebra and trigonometry provide us with a proof that the general angle can not be trisected using just a straight edge and compass as follows:

If we consider an angle of 60° and use the triple angle formula for cosine,

$$\cos 3x = 4 \cos^3 x - 3 \cos x,$$

we get the following equation letting z = x/3 and noting that cos 60° = 1/2:

$$4x^3 - 3x - 1/2 = 0,$$

or

$$8x^3 - 6x - 1 = 0. \qquad (2)$$

Recall that Euclidean tools construct only circles and straight lines. The simultaneous solution of equations for circles and straight lines with rational coefficients can only have roots that are rational or of the form $a + b\sqrt{c}$ where a, b and c are rational numbers. Now, a rational root of equation (2) must have the form a/b where a is a factor of 1 and b is a factor of 8. By using synthetic division and trying all the possibilities, we find that equation (2) has none. If $a + b\sqrt{c}$ is a root of equation (2), then $a - b\sqrt{c}$ must also be a root. If we let r be the third root of equation (2), the $r + (a + b\sqrt{c}) + (a - b\sqrt{c}) = 0$ since the coefficient of the $x^2$ term of equation (2) is zero. This means that $r + 2a = 0$, so $r = -2a$ which is a rational number, a contradiction of the above results. Thus equation (2) can not have roots of the form $a + b\sqrt{c}$. Therefore, the roots of equation (2) are not constructible, so we can not trisect a 60-degree angle in particular which proves that we can not trisect every angle with Euclidian tools.

Thus we see that the study of trisecting angles is interesting. People in general and students in geometry classes in particular will probably continue to try to trisect angles. Even though they must fail in the attempt, the exercise in thinking and reasoning may be good for them because, throughout history, mathematicians and other people exploring the unknown have often learned more from their mistakes than from their successes.

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