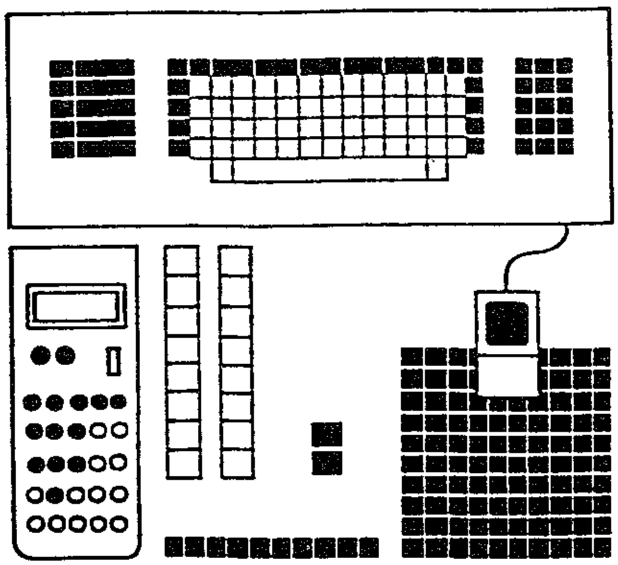
# TEXAS MATHEMATICS TEACHER



**STUFF "Strategic Tactics Ultimately For Fun"**

**A Characterization for Conic Sections**

**Surveys and Sampling Errors**

**Useful Patterns**

**TCTM Candidate Ballot**

## JANUARY 1989

**TEXAS MATHEMATICS TEACHER**, the official journal of the Texas Council of Teachers of Mathematics, is published four times each year, in October, January, March, and May. All manuscripts and editorial correspondence should be addressed to George H. Willson, listed above.

Manuscripts should be neatly typewritten and double spaced, with wide margins, on 8 1/2 " by 11" paper. Authors should submit the signed original and four copies. Illustrations should be carefully prepared in black ink on separate sheets, the original without lettering and one copy with lettering added. As soon as possible after refereeing, authors will be notified of a publication decision. Originals of articles not accepted for publication will be returned to the authors. Two copies of the issue in which an author's manuscript appears will automatically be sent to the author.

SUBSCRIPTION and MEMBERSHIP information will be found on the back cover.

# TEXAS MATHEMATICS TEACHER

TEXAS COUNCIL OF TEACHERS OF MATHEMATICS

**Affiliated with the National Council of Teachers of Mathematics**

**January 1989**

# TABLE OF CONTENTS

## ARTICLES



## ANNOUNCEMENTS

## President's Message

### HELP WANTED !!!!
### GOOD TEXAS-AMERICAN MATHEMATICS TEACHERS!!!!

The State of Texas, in fact, the United States as a whole, needs a steady supply of students dedicated to the teaching of mathematics and its applications. Fewer Americans are interested each year in studying advanced mathematics. In the past academic year, approximately 400 Americans received doctorates in mathematics and this only represented 49% of the total doctorates in mathematics. The others went to non-American students. Most teaching fellows at the colleges and universities are non-Americans.

We, who are at the helm now, must find a way to stem the tide; reverse the trend, and stop the decline in the number and quality of mathematic students in this country.

We are the leaders in mathematics education and must work together with our fellow teachers in the elementary schools. It is here that the likes and interests of the students are molded. If most students do not have an interest in mathematics by the time that they are in the sixth grade, then they are usually lost to mathematics forever.

Texas can become the great leader in turning this around through the help and cooperation of industry and education.

NCTM has written a great set of standards for a national curriculum. Let's be the first to use these and make the most of them in arousing our students to succeed in mathematics and its applications. Let's learn these standards; apply them and help others to use them. No, they are not the complete answer to all of our problems; they are not the end result; but they are a great start.

The Texas TEAMS is a minimum, not a maximum, as many school districts seem to think. We must strive to push our students well beyond these small goals. We must work on the higher order thinking skills. We must work on problem solving, reading comprehension, modeling skills, and incorporate modern technology into our everyday curriculum and teaching.

This is a battle that we must win. This is a battle that we must recruit others to join. It is not hopeless, but it will be hard.

Many major universities are beginning to help us as they require more mathematics and science of business majors and more business courses of the technical students. This blend will help students understand that they need all subjects to succeed.

Let's all join together and march shoulder to shoulder as stout-hearted men and women. We need more young dedicated teachers to enter the teaching profession. It is now that we must work together, not later.

OTTO W. BIELSS, JR.

**Note From the Editors**

Please check the expiration date on your mailing label, your address and zip code. If any are incorrect, send your correction to John Huber, Sam Houston State University, P. O. Box 6768, Huntsville, Texas 77340. The data base for TCTM is kept by John.

When submitting journal articles, please send the signed original and four copies. Figures should be drawn on separate sheets and in black ink. The masthead page of the journal has complete directions.

## TEXAS COUNCIL OF TEACHERS OF MATHEMATICS
## CANDIDATE INFORMATION

**VICE-PRESIDENT** (representing middle school)

| Candidate | Position | Organizations, Current Offices |
|---|---|---|
| Beverly Millican | Elementary Coordinator, Plano ISD, (Region X, K-12 until this fall) | NCTM, TCTM, TASM (TX Assoc. Super. Math) Editor of STUFF column for TCTM Journal, STEAM Regional Representative |
| Dorothy Ware | Math Supervisor, K-12 Galena Park ISD | NCTM, TCTM, SJACTM, TASM, NCSM, Past Pres. TASM, Editor of STEAM Newsletter |

**SECRETARY**

| Candidate | Position | Organizations, Current Offices |
|---|---|---|
| John Huber | Professor of Math and Computer Education SHSU | NCTM, TCTM, TBRVCTM, TASM, TCEA, MAA, Secretary of TCTM |
| Sandra Ingram | Math D C Kingwood High School | NCTM, TCTM, 1960 Area CTM, MAA |

---

# BALLOT

**Vice-President**

**___ Beverly Millican**
**___ Dorothy Ware**

**Secretary**

**___ John Huber**
**___ Sandra Ingram**

**Vote for one candidate for each office. Mail to: George Willson**
**2920 Bristol**
**Denton, TX 76201**

**<u>Postmark by February 15, 1989</u>**

# STUFF

## ***Strategic Tactics Ultimately For Fun***

**Beverly Millican**

*Elementary Mathematics Coordinator*
*Plano Independent School District*

Even though the '88 Olympics are now memories, teachers can perpetuate this competitive spirit in their classrooms by using game format activities, such as the following "Think Olympics."

Lyn Englebert, a 4th grade teacher in Plano ISD, has adapted a team activity from a "Mathletic"event that she and other 4th grade teachers used when teaching in Alief ISD. While credit is given for the correct answer in this game, the emphasis is still on the process(es) involved, since a great deal of time is spent discussing various "plans of attack" after the problems have been solved.

In the following sample activities, 5th grade level problems are given. One of the advantages of this kind of format is that it can be adapted to a variety of ability or grade levels and still be active and dynamic learning!

### MATH OLYMPICS

A. **Objective:** To give children practice in solving non-routine problems in a fun, motivating format.

B. **Preparation:**

1. Arrange students in teams of 4 (maximum). Try to get an even mix of ability levels. Select a team captain.

2. Make an envelope or folder for each team. Label each with the team number.

3. Make a copy of each problem to insert in each team's folder.

4. Make transparency of each problem.

5. Put team numbers and problem names on the scoreboard. See attached scoreboard sheet.

6. Make awards for 1st, 2nd, and 3rd place teams.

C. **Procedure**

1. Bring children to the front of the room to explain what Math Olympics is and how it will be run.

2. Explain rules.

   a. Each team has a team captain. Only the captain may bring up an answer to be checked.
   b. If the answer is correct, the team's points are recorded on the scoreboard. If the answer is wrong, the scorekeeper will subtract one point from the total possible points for that answer and write the new total on the team's answer sheet. Each time an incorrect answer is brought up, the total possible points for the problem in question is decreased by one point.
   c. The team may only work on one at a time. The team may look over all the problems, but should decide which problems to solve and return the other sheets to their envelope. This rule encourages teamwork.
   d. Bonus points may be awarded for teams which work well together. Children should be made aware that they should discuss the problems and answers thoroughlywithin their groups.

3. With children sitting in their teams, pass out team envelopes and read through the problems.

4. Send teams to a work space with their envelopes, paper, and pencil.

5. Allow 30 minutes for the teams to work on the problems.

6. If a team finishes early, give them one of the problems they have solved and have them plan a brief presentation to explain the answer during the class strategy session.

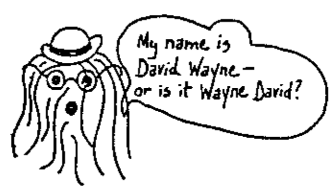
7. Total the points on the scoreboard and give out awards.

D. **Strategy Session:** This is a very important part of the problem solving experience. Go over each problem, allowing children to explain how they solved the problems. Many approaches to the solutions should become apparent and it is very beneficial to discuss all of them. (Allow ample time for this part of a Math Olympics.)

Team ______
Points __5__

**First and Last Names**

The last names of Helen, Irving and Jacqueline are Abrams, Barrow and Clancy, but not necessarily in that order. Clancy is Jacqueline's uncle. Helen's last name is not Barrow. What are each person's first and last names?




Team ______
Points __8__

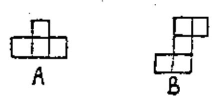
**What Are Your Chances?**

Suppose that you roll two triangular pyramids, one of which has faces numbered 0 through 3, and the other numbered 1 through 4. What are your chances of rolling a sum of 5?

Team ______
Points 9

### The Perimeter Problem

Each of the small boxes in these figures is a square. All the squares are the same size. If the perimeter of figure A is 30, what is the perimeter of figure B?




Team ______
Points 9

### Crack The Code

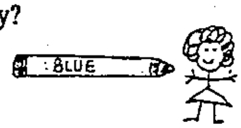
In the multiplication problem shown, each letter represents a different digit. If A is not zero, what are the values of A, B, C, and D?

| | | |
|---|---|---|
| ABC | A = | C = |
| x C | B = | D = |
| DBC | | |

Team ______
Points 7

### The Crayon Problem

At the end of one school day, a teacher had 17 crayons left. The teacher remembered giving out 14 crayons in the morning, getting 12 crayons back at recess, and giving out 11 crayons after lunch. How many crayons did the teacher have at the start of the day?



Team ______
Points __11__

**A Resolution That Makes Cents!**

Your New Year's Resolution is to put 1 cent into your savings account the first day of the year, 2 cents the second day, 3 cents on the third day, 4 cents on the fourth day, and so on for all 365 days of the year. How much money (in dollars and cents) will you have put in your savings account by the end of the year?

**Answer Key**

First and Last Names:
Helen Abrams — *Irving is only male name, so he must be Jacqueline's uncle.
Irving Clancy
Jacqueline Barrow

What Are Your Chances: 3 out of 16

The Perimeter Problem: Figure B = 36

Crack the Code:

| | | |
|---|---|---|
| A = 1 | | A = 1 |
| B = 2 | or | B = 7 |
| C = 5 | | C = 5 |
| D = 6 | | D = 8 |

The Crayon Problem: 30 crayons

Resolution That Makes Cents:
$667.95 — *182 pairs of 366¢ plus 183¢ (middle no.)

# A CHARACTERIZATION FOR CONIC SECTIONS

**James R. Boone**

*Texas A&M University*
*College Station, Texas*

A characterization of conic sections is presented in this brief note. Construction activities, enrichment exercises requiring basic properties of conic sections and applications for further study are included. A lesson preparation for a pre-calculus class precipitated the observation that the fixed difference [sum], say r, of the distances in the definition of a hyperbola [ellipse] determines a generating circle, C, (of radius r about one focus, $F_1$) which with the other focus $F_2$ generates the points on the hyperbola [ellipse] in a natural way as the set of points which are equidistant from C and $F_2$ [Fig. 1]. Accordingly, the characterization can now be stated.

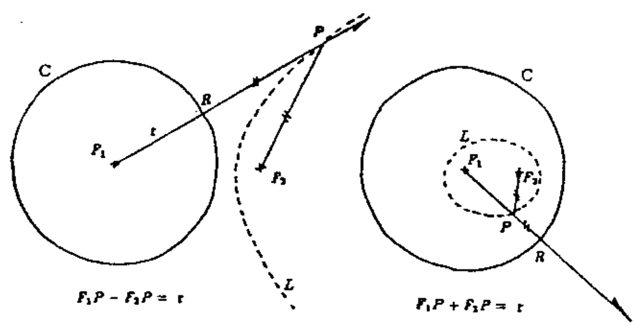



Fig. 1

CHARACTERIZATION: Every hyperbola (ellipse) is precisely the set of points which are equidistant from some circle and some point outside (inside) the circle.

An activity which may reinforce basic properties of circles, isosceles triangles, hyperbolas and ellipses would be to construct several points on the hyperbola [ellipse], L, generated by circle C centered at $F_1$ of radius r and another point $F_2$ outside [inside] the circle C as follows: Draw a point $F_2$ and the circle of radius r, centered at $F_1$. If R is on the circle, then the perpendicular bisector of $\overline{RF_2}$ intersects $\overrightarrow{F_1R}$ at a point on the hyperbola [ellipse] generated. The converse of this exercise, which would be easy to grade, would be to strike the generating circle, given the foci and one point on a conic L. One will observe that there is both a unique hyperbola and a unique ellipse which satisfy these conditions.

Another more direct construction, which does not use perpendicular bisectors, of points on the hyperbola [ellipse], given $F_1$,$F_2$ and the radius r, would be to strike a circle of radius t about $F_2$, then strike a circle of radius r + t [radius r - t] about $F_1$. One will quickly find that t needs to be large enough for the circles to intersect. The two points where these circles intersect, if any, are on the hyperbola [ellipse].

The connection between this generating circle approach and the standard form of a hyperbola symmetric about the origin can easily be shown [Fig. 2], where a and b are the lengths of the semi-major and semi-minor axes, $r = 2a$, $c^2 = a^2 + b^2$, and $2c = F_1F_2$.

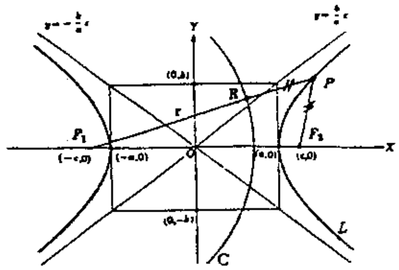



Fig. 2

In the case of the hyperbola, it is an instructive problem solving experience to establish that the point R on the generating circle C cannot be such that $m\angle RF_1F_2 = \tan^{-1}(\frac{b}{a})$, because $\frac{b}{a}$ is the slope of the asymptotes. Similarly, if R is a point on C such that $m\angle RF_1F_2 > \tan^{-1}(\frac{b}{a})$, no points on the hyperbola are determined. Also, when using points R where $m\angle RF_1F_2 < \tan^{-1}(\frac{b}{a})$, points are determined on the branch near $F_2$.

The eccentricity is given by $e = \frac{c}{a}$ and thus $e = \frac{F_1F_2}{r}$. This characterization then gives a new view of the well-known eccentricity implications as follows:

a) If $e = 0$, then $c = 0$ and $F_1F_2 = 0$. Since $F_2 = F_1$, the circle centered at $F_1$ of radius $\frac{r}{2}$ is the conic.

b) If $0 < e < 1$, then $c < a$ and $F_1F_2 < r$. This implies $F_2$ is inside the generating circle and an ellipse is the conic.

c) If $e = 1$, then $F_1F_2 = r$. Any line can be considered as a circle of infinite radius whose center $F_1$ is at the point at infinity. Since $e = \lim_{F_1 \to \infty} \frac{F_1F_2}{r} = 1$, we have the consistent result that the parabolas are the sets of points which are equidistant from a focus $F_2$ and a line (generating circle centered at infinity having infinite radius).

d) If $e > 1$, then $c > a$ and $F_1F_2 > r$. This implies $F_2$ is outside the generating circle and a hyperbola is formed.

The full spectrum of changes in conic sections, related to increasing eccentricity, may be considered [Fig. 3]. Let $F_1$ be a fixed point and let C be any fixed circle of radius r centered at $F_1$. We can view the metamorphosis of the conics as the other focus $F_2$ is allowed to migrate from the initial position, $F_2 = F_1$, to infinity. When $F_2 = F_1 (e = 0)$ the conic L is the circle centered at $F_1$ of radius $\frac{r}{2}$. As $F_2$ proceeds toward the rim of C the ellipse elongates until $F_2$ reaches C. When $F_2$ touches C($e = 1$), a geometric discontinuity occurs when the ellipse can be flattened no more and it snaps like a rubber band. It is as if the radius momentarily becomes infinite, generating a parabola at that instant only. Immediately following this break at $e = 1$, for $e > 1$ the set of points L reassembles on the outside of C as a hyperbola which goes from shallow to sharp as e becomes large without bound. In Fig. 3, only one branch of each hyperbola is shown.

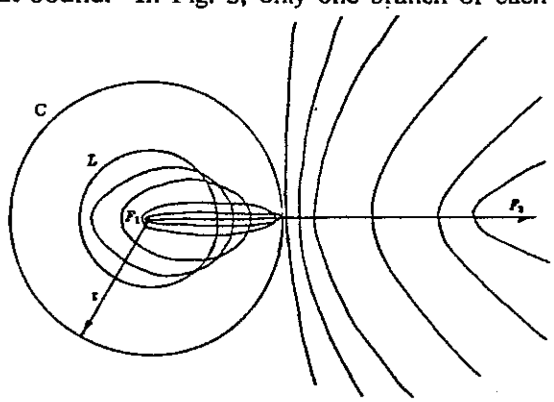



Fig. 3

Further, since $e = \frac{F_1F_2}{r}$, the hyperbola and the ellipse that share the same foci $F_1$ and $F_2$ and a common point P on both conics, have concentric generating circles centered at $F_1$. If these circles have radii $r_H$ and $r_E$ and eccentricities $e_H$ and $e_E$, then $e_H r_H = e_E r_E$ (= $F_1F_2$).

Additional occurrences of hyperbolas come to mind from the characterization. I will close with two personal favorites which can be observed in nature. The projection, to the ground, of the curve of intersection of right circular cones of sand, (with different altitudes and with sand of equal coarseness for equal angles at the apex) is a part of a branch of a hyperbola. The verification of this fact provides an exercise in three-dimensional visualization, angle-side-angle and congruence applications [Fig. 4].

The second occurrence comes from waves. If we throw a pebble in a still pond, and later throw another pebble striking outside the expanding ripple of the first, the set of points at which the two ripples meet is the branch of a hyperbola. If the pond is a circle (which will reflect ripples) and the first pebble is thrown to the center, and the second pebble is thrown to strike inside the ripple of the first, then the set of points where the ripple of the second pebble intersects the reflected ripple of the first pebble is an ellipse. I am assuming that the ripples travel at a constant radial velocity. Students might also study applications in the loran navigational system and the similar method for determining artillery firing positions by the use of firing time differences as heard by two (or more) field artillery forward observers.

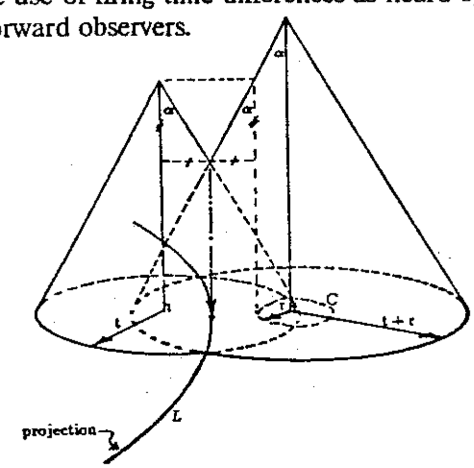



Fig. 4

# SURVEYS AND SAMPLING ERRORS

**Joe Dan Austin**

*Rice University*
*Houston, Texas*

Perhaps the most uniquely American "institution"or activity is the public opinion survey. There are surveys for political issues, pre-election predictions, post-election analyses, product evaluation and most aspects of daily life. A typical report is the following:

> In a survey of 600 likely voters, 34% planned to vote for the incumbent, 20% planned to vote for challenger A, 14% planned to vote for challenger B, and the rest were undecided. The survey has an error of plus or minus 4%.

Most students easily understand how to find the percent of voters favoring or opposing a candidate (or issue) from the survey data. However, few students understand how the "error of plus or minus 4%" is computed or exactly what it means. An explanation and method of computing this number will be presented.

The "error of plus or minus 4%" reflects possible sampling error and means that the (unknown) percent of all voters favoring the incumbent should be in the interval 34 - 4 percent (30%) to 34 + 4 percent (38%). (Of course, this unknown percent is seldom known except when all people are surveyed as in the actual election.) Other samples of likely voters may give different percents favoring the incumbent and different intervals. Some of these intervals will contain the unknown percent favoring the incumbent while some will not. The fraction of intervals that contain the unknown percent can then be computed. This fraction suggests how confident we are that the interval reported after one survey does contain this unknown percent. (As usual, this fraction or probability is not given in the report but is usually 0.95 or 0.99.) The interval reported is called a "confidence interval."

To understand how to derive a confidence interval for a survey, a probability model is needed. Let p be the unknown fraction of all voters who favor the incumbent. The fraction favoring the incumbent (or issue) is always unknown. In fact, the main purpose of the survey is to estimate p. Let n be the number of likely voters in a sample, and let g be the fraction of people in the sample who favor the incumbent. (In the example, $n = 600$ and $g = 0.34$.)

To find a 95% confidence interval we want to find c, so that

(1) $P[g - c \leq p \leq g + c]$

is 0.95 (or larger). (In the example, $c = 0.04$.)

If the sample is small (n is 5% or less of all voters), the (standardized) random variable

(2) $$Z = \frac{g - p}{\sqrt{p(1-p)/n}}$$

is approximately normal with mean 0 and variance 1. (This is a form of the Central Limit Theorem. For example, see Freund, 1984, §7.4 and §8.5.) Since $|p - q| \leq c$ is equivalent to $|g - p| \leq c$, the probability in (1) can be rewritten to use (2) and thus gives

$$\begin{aligned} P[g - c \leq p \leq g + c] &= P[-c \leq g - p \leq c] \\ &= P[-c\sqrt{n}/\sqrt{p(1-p)} \leq Z \leq c\sqrt{n}/\sqrt{p(1-p)}] \end{aligned}$$

To find c, consider the last probability. As the square of a number is non-negative.

$$p(1-p) = 1/4 - (1/2-p)^2 \leq 1/4.$$

Thus $1/p(1-p) \geq 4$, and $1/\sqrt{p(1-p)} \geq 2$. Also $-1/\sqrt{p(1-p)} \leq -2$. Therefore,

(3) $P[-c\sqrt{n}/\sqrt{p(1-p)} \leq Z \leq c\sqrt{n}/\sqrt{p(1-p)}] \geq P[-2c\sqrt{n} \leq Z \leq 2c\sqrt{n}]$.

(The inequality between the probabilities in (3) follows from the fact that if A  B then $P[A] \leq P[B]$.) When the latter probability in (3) is greater than or equal to 0.95, then the probability in (1) will also be greater than or equal to 0.95. As Z is approximately normal, a normal probability distribution table gives that, for a probability of 0.95, Z is between +1.96 and -1.96. See Figure 1.

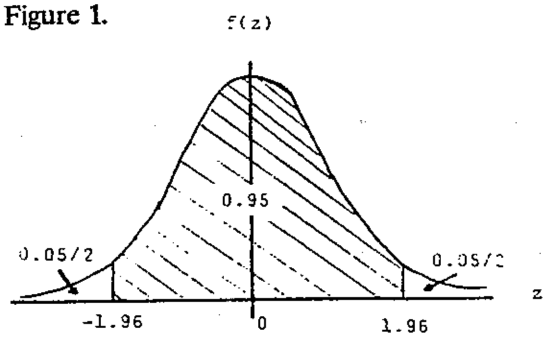



Fig. 1. Standard Normal Density with 0.95 probability interval about 0

For a 95% confidence interval

(4) $2c\sqrt{n} = 1.960$, so $c = 0.98/\sqrt{n}$.

For a 99% confidence interval, similar reasoning gives

(5) $2c\sqrt{n} = 2.576$, so $c = 1.288/\sqrt{n}$.

Using $n = 600$ in (4) gives $c = 0.04$. Using $n = 600$ in (5) gives $c = 0.05$. (The higher the probability for the confidence interval, the wider the confidence interval.) In the election example, the confidence interval is a 95% confidence interval.

For a given probability, the sample error c depends only on the sample size n. Therefore, in the election example, not only does the percent of voters for the incumbent have a sampling error of ±4%, but the percent of voters favoring each of the challengers also has the same error. For example, the percent of voters favoring challenger B has a 95% confidence interval of [10%, 18%] or ±4%. Similarly, a 95% confidence interval for the percent of undecided voters is [28%, 36%] as the percent of undecided voters is 100% minus 68% (34% + 14% + 20%) or 32%.

When you read survey results, compare the survey errors with those obtained using (4) and (5). You might also find it interesting to simulate a survey using a computer where you set the percent of the population that favors the issue. In your simulations, how often do the 95% confidence intervals include this percent? Does this percent depend on the percent of the population that favors the issue? Does it depend on the sample size?

Reference

Freund, John E. Modern Elementary Statistics (6th ed.). Englewood Cliffs, N.J.: Prentice-Hall, 1984.

REFEREES WANTED

Manuscripts published in the Texas Mathematics Teacher are reviewed by at least three mathematics educators. Members of TCTM who are regular readers of the journal - classroom teachers, supervisors, and teacher educators - who would like to review manuscripts should write to the TCTM indicating their willingness to serve and the level of interest (Elementary, Secondary, or both). Contact George H. Willson, Box 13857, University of North Texas, Denton, Texas 76203-3857. The Editorial Panel will review the responses and make the final selection.

# USEFUL PATTERNS

**Andrew Scott and William Hastings**

*Department of Mathematics*
*Berea College*
*Berea, Kentucky*

Most school children are familiar with addition tables of the type shown in Fig. 1.

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | | |
| 1 | | | | | | | | | | |
| 2 | | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | | | | | | | | | |
| 5 | | | | | | | | | | |
| 6 | | | | | | | | | | |
| 7 | | | | | | | | | | |
| 8 | | | | | | | | | | |
| 9 | | | | | | | | | | |

Fig. 1

The exercise is valuable in that it helps pupils in learning or reviewing addition facts and also introduces them to several interesting and useful patterns. When filling out the table, students should be urged to work neatly and place well-formed numerals in the center of the individual squares. Be permitting students to work in pairs or in small groups careless errors will be minimized. Both while entering numerals and after completion of the table, students should be encouraged to observe forming patterns.

Many patterns may be observed within the table to which children exhibit varying degrees of sensitivity. One highly visible pattern that may be recognized by many students is that the numbers in column 1 are the same as those in row 1. This pattern holds for comparable columns and rows. A less obvious pattern often noted by fewer students is that the numbers in the rising diagonals always differ from their neighbors along the diagonal by 2.

Patterns within patterns also can be discovered. To observe this, construct a 2 by 2 mask and place it on the addition table exposing the numbers shown in Fig. 2.

| | |
|---|---|
| 7 | 8 |
| 8 | 9 |

Fig. 2

The sum of the two numbers on the rising diagonal, $8 + 8 = 16$, is the same as the sum of the two numbers on the falling diagonal, $7 + 9 = 16$. The pattern holds throughout the table, regardless of the placement of the mask.

Should a 3 by 3 mask be used, the sums of the three numbers on either diagonal are alike as in fig. 3. Likewise for a 4 by 4 mask as shown in Fig. 4.

| | | |
|---|---|---|
| 3 | 4 | 5 |
| 4 | 5 | 6 |
| 5 | 6 | 7 |

Fig. 3

| | | | |
|---|---|---|---|
| 2 | 3 | 4 | 5 |
| 3 | 4 | 5 | 6 |
| 4 | 5 | 6 | 7 |
| 5 | 6 | 7 | 8 |

Fig. 4

The addition table also serves as a subtraction table demonstrating the Commutative Property as shown in Figs. 5 and 6.

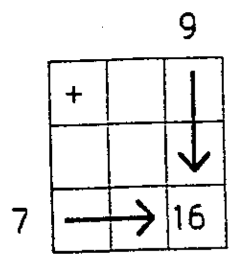



Fig. 5

Addition
7 + 9 = 16
9 + 7 = 16

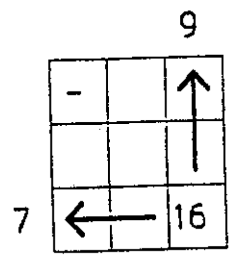



Fig. 6

Subtraction
16 - 7 = 9
16 - 9 = 9

The addition table can serve as a pre-introduction to the great subject of series. As an example, the numbers in Row 1 of the table can laboriously be added one by one or more easily summed by using a readily remembered formula. Utilizing the formula both simplifies calculation and reduces opportunity for error.

The useful formula is: $$\frac{(F + L)(L - F + d)}{(2)(d)}$$

Where F represents the First number of the series.
and L represents the Last number in the series.
and d represents the difference between adjacent numbers.

For example, to add the numbers on line 1 which are

1 2 3 4 5 6 7 8 9

substitute the proper numbers, F = 1, L = 9, d = 1, in the formula to obtain

$$\left(\frac{1 + 9}{2}\right)\left(\frac{9 - 1 + 1}{1}\right) = 45$$

Similarly, the sum of the numbers on line 7

7 8 9 10 11 12 13 14 15 16 17

can readily be found. To fix the formula firmly in mind. The teacher may assign the following as well as similar arithmetic series to students for calculation.

$1 + 2 + 3 + 4 + 5$
$5 + 10 + 15 + 20 + 25 + \ldots + 100$
$10 + 8 + 6 + 4 + 2$
$1 + 1\frac{1}{2} + 2 + 2\frac{1}{2} + 3$

A discussion of patterns lends consideration to Magic Squares. Observe Fig. 7 which is a third order magic square. To construct it use the positive integers 1 through 9 without duplication and it will be seen that the sum of the numbers in any column, row or diagonal is 15.

The fourth order magic square, Fig. 8, was created by the German artist Albrecht Durer (1471 - 1528) for use in his famous engraving *Melanchola*. Here the numbers 1 through 16 are used; columns, rows and diagonals total 34.

| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

Fig. 7

| | | | |
|---|---|---|---|
| 16 | 3 | 2 | 13 |
| 5 | 10 | 11 | 8 |
| 9 | 6 | 7 | 12 |
| 4 | 15 | 14 | 1 |

Fig. 8

The renowned Benjamin Franklin (1706 - 1790) enjoyed constructing magic squares of the tenth order. Both Dürer and Franklin knew that if rows were changed to columns or columns to rows, in the same order, the square remained magical as demonstrated in Figs. 9 and 10.

| | | |
|---|---|---|
| 4 | 3 | 8 |
| 9 | 5 | 1 |
| 8 | 7 | 6 |

Fig. 9

| | | | |
|---|---|---|---|
| 5 | 10 | 11 | 8 |
| 16 | 3 | 2 | 13 |
| 4 | 15 | 14 | 1 |
| 9 | 6 | 7 | 12 |

Fig. 10

## MULTIPLICATION TABLE PATTERNS

having discovered many significant patterns in the addition table, it will be much easier to complete the multiplication table and discover several of its nice regularities and interesting patterns.

| X | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | | |
| 1 | | | | | | | | | | |
| 2 | | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | | | | | | | | | |
| 5 | | | | | | | | | | |
| 6 | | | | | | | | | | |
| 7 | | | | | | | | | | |
| 8 | | | | | | | | | | |
| 9 | | | | | | | | | | |

Fig. 11

Again, students should be encouraged to work carefully, to place well-formed numerals in the center of the cells and most importantly, to check answers with classmates and correct any errors that have been made. In this manner, as patterns emerge, students may avoid frequent errors.

There are numerous patterns to be found in the multiplication table and the following questions will assist students in finding some that may be overlooked.

- Are the numbers in row 1 the same as those in column 1?
- Does the pattern hold for comparable rows and columns?
- Can patterns be found on the rising diagonals? On the falling diagonals? (See hint at the end of this article.)
- If a 2 by 2 mask is placed on the multiplication table, are the diagonal products alike? For a 3 by 3 mask?
- Can the multiplication table serve as a division table?
- Can the numbers in any row or column be summed by a readily remembered formula?

The numbers in column 9 are:

9
18
27
36
45
54
63
72
81

Can you find two patterns in this set of numbers?

The sums of the numbers on the rising diagonals are:

| Diagonal | Sum |
|---|---|
| 1 | 1 |
| 2 | 4 |
| 3 | 10 |
| 4 | 20 |
| 5 | 35 |
| 6 | 56 |
| 7 | 84 |
| 8 | 120 |
| 9 | 165 |

Question: Can this set of numbers be found, in that order, in Pascal's Triangle?

The multiplication table may also be used to generate sets of common fractions which, after appropriate reduction, exhibit many interesting patterns. Let us see.

If the numbers in Row 1 are used as numerators and those in row 2 as denominators, are the rational numbers so produced equivalent fractions?

$$\frac{\text{Row 1}}{\text{Row 2}} \quad \frac{1}{2} \quad \frac{2}{4} \quad \frac{3}{6} \quad \frac{4}{8} \quad \frac{5}{10} \quad \frac{6}{12} \quad \frac{7}{14} \quad \frac{8}{16} \quad \frac{9}{18}$$

* * * *

If the numbers on Rising Diagonal 6 are numerators and those on Rising Diagonal 7 are denominators, what patterns result?

$$\frac{\text{Rising Diagonal 6}}{\text{Rising Diagonal 7}} \quad \frac{6}{7} \quad \frac{10}{12} \quad \frac{12}{15} \quad \frac{12}{16} \quad \frac{10}{15} \quad \frac{6}{12}$$

* * * *

If the numbers on Falling Diagonal 1 are numerators and those on Falling Diagonal 3 are denominators, what patterns, after appropriate reduction, can be identified?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Falling Diagonal 1 | 1 | 4 | 9 | 16 | 25 | 36 | 49 |
| Falling Diagonal 3 | 3 | 8 | 15 | 24 | 35 | 48 | 63 |

* * * *

If the numbers on horizontal Row 6 are numerators and those on Rising diagonal 6 are denominators, what patterns emerge?

| | | | | | | |
|---|---|---|---|---|---|---|
| Horizontal Row 6 | 6 | 12 | 18 | 24 | 30 | 36 |
| Falling Diagonal 6 | 6 | 10 | 12 | 12 | 10 | 6 |
| | | | | | | |
| | 1 | 6 | 3 | 2 | 3 | 6 |
| | 1 | 5 | 2 | 1 | 1 | 1 |
| | | | | | | |
| | 6 | 6 | 6 | 6 | 6 | 6 |
| | 6 | 5 | 4 | 3 | 2 | 1 |

Here the search for a pattern is a bit more difficult but a feeling for form or a happy inspiration leads to the discovery of the pattern.

Other interesting number patterns will be discovered by working the exercises given below.

| | |
|---|---|
| Horizontal Row 5 | Vertical Column 5 |
| Horizontal Row 8 | Vertical Column 9 |
| | |
| Rising Diagonal 6 | Falling Diagonal 2 |
| Falling Diagonal 7 | Rising Diagonal 8 |
| | |
| Falling Diagonal 2 | Falling Diagonal 2 |
| Rising Diagonal 9 | Rising Diagonal 8 |

Students may choose to create their own problems, solve them and find the patterns, or they may construct challenge problems for their classmates to solve and discover the nice patterns which are sometimes "hidden."

Patterns are always significant and this is especially true in mathematics. For example, the East Indian poet, Pingala, discovered the Binomial theorem in 200 B.C. through his studies of patterns in poetry. Pascal and

Fermat discovered probability and combinatorics by assisting the Chevalier de Mere, a gambler, to discover winning patterns in games of chance. The Austrian monk, Gregor Mendel, in his botanical studies found patterns that established plant and animal genetics as a science. More recently, it was the recognition and appreciation of the significance of patterns that led to the controlled release of atomic energy.

Patterns and what they reveal will continue to engender great advances in the physical, biological and social sciences as well as the humanities. Accordingly, at an early age, students should be urged to be alert for patterns, to search for them and pursue them relentlessly.

It is interesting to speculate that all mathematics is but a refinement and extension of the patterns revealed by the four fundamental operations of arithmetic.

Hint: Consider Falling Diagonal 3.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| The numbers are: | 3 | | 8 | | 15 | | 24 | | 35 | | 48 | | 63 |
| | | V | | V | | V | | V | | V | | V | |
| First differences: | | 5 | | 7 | | 9 | | 11 | | 13 | | 15 | |
| | | | V | | V | | V | | V | | V | | |
| Second differences: | | | 2 | | 2 | | 2 | | 2 | | 2 | | |

From the foregoing display, algebra students may be able to derive the formula $N^2 + 4N + 3$ which produces the numbers on Falling Diagonal 3, (3, 8, 15, 24, 35, 48, . . .) as N runs through the values 0, 1, 2, 3, . . . .

# PERSPECTIVE ON TESTING
## by Ross Taylor

*Mathematics Consultant*
*Minneapolis Public Schools*
*Minneapolis, Minnesota*

Students who finished kindergarten in 1988
Plan in the year 2000 to graduate.
This fact makes it clear to you and me
That we are preparing students for the next century.

The world they will face will demand
That they think critically and understand.
They won't need rote learning of computation,
They will need problem solving and estimation.

Today adults compute by electronic means
And so should children as well as teens.
The computation that they will need later
Will be done by computer and calculator.

They will also need to gain facility
With topics like statistics and probability.
They will need skill in reasoning and communications
And knowledge of patterns, functions and relations.

We see that changes are needed right away.
The changes can't wait. We must begin today.
If we clearly define the what and the how,
Teachers can start to make changes now.

We recognize that the changes that need to be
Are influenced by three factors that begin with T;
Teachers, textbooks and tests are the keys
To make the changes that we please.

Textbooks need not prevent the change;
Teachers can skip, supplement and rearrange.
However, when tests are standardized,
They cannot be altered or revised.

The curriculum and the tests should be aligned
With new goals and objectives that are defined.
To ensure sound curriculum practices are heeded,
It is clear that new standardized tests are needed.

Tests to assess the next generation
Must not focus on computation.
Doing mathematics with a calculator is the best,
So we should allow calculator use on a test.

We cannot make our progress complete
Using tests that are obsolete.
Tests take years to develop and to norm,
So they could be a barrier to reform.

The use of today's standardized tests should halt,
Or we should take the results with a big grain of salt.
By placing testing in proper perspective,
We can make instruction relevant and effective.

## TEXAS COUNCIL OF TEACHERS OF MATHEMATICS

Affiliated with the National Council
of Teachers of Mathematics
1988-89

**PRESIDENT:**
Otto Bielss, 2607 Trinity Street, Irving, TX 75062

**VICE-PRESIDENTS:**
Cathy Rahlfs, Humble ISD, P. O. Box 2000, Humble, TX 77347
Susan M. Smith, Ysleta ISD, Ysleta, TX 79907
Beverly R. Cunningham, Rt. 6, Box 1645 A, Bulverde, TX 78163

**SECRETARY:**
John Huber, Box 2206, Huntsville, TX 77341

**TREASURER:**
Sally Lewis, P. O. Box 33633, San Antonio, TX 78265

**NTCM REPRESENTATIVE:**
Bill Duncker, 702 North N. Street, Midland, TX 79701

**REGIONAL DIRECTORS OF T.C.T.M.:**
Elgin Schilhab, 2305 Greenlee, Austin, TX 78703
Bob Mora, 2517 Lawnview Drive, Carrollton, TX 75006
Charles Reinauer, 3704 Longwood, Pasadena, TX 77503
Sally Ann Rucker, P. O. Box 7862, Midland, TX 79708

**PARLIAMENTARIAN:**
Marilyn Rindfuss, 109 Laburnum Drive, San Antonio, TX 78209

**JOURNAL EDITOR:**
George H. Willson, P. O. Box 13857, University of North Texas, Denton, TX 76203-3857

**CO-EDITOR:**
James Bezdek, P. O. Box 13857, University of North Texas, Denton, TX 76203

**TEA CONSULTANT:**
Cathy Peavler, Director of Mathematics, 1701 Congress, Austin, TX 78701

**NCTM REGIONAL SERVICES:**
Suzanne Mitchell, 432 N. E. Churchill St., Lees Summit, MO 64063, (501)490-2000

TEXAS MATHEMATICS TEACHER
George H. Willson, Editor
Texas Council of
Teachers of Mathematics
P. O. Box 13857
University of North Texas
Denton, TX 76203-3857

NON-PROFIT
ORGANIZATION
U. S. Postage
PAID
Dallas, Texas
Permit #4899

MEMBERSHIP
Texas Council of Teachers of Mathematics
Affiliated with the National Council of
Teachers of Mathematics
Annual Membership Dues for Teachers $8.00
USE THIS CARD FOR MEMBERSHIP

Cut on dotted line

Texas Council of Teachers of Mathematics

| Last Name | First Name | School | (Leave Blank) |
|---|---|---|---|
| **Street Address** | **City** | **State** | **Zip** |

Dear Teacher,
To ensure continuous membership, please print your name, zip code, and school above. Enclose this card with your check for $8.00 for 1 year payable to T.C.T.M. and mail to:

Sally Lewis
Treasurer
P. O. Box 33633
San Antonio, TX 78265

____ Renewal ____ New ____ Change of Address

Circle area(s) of interest: K-2 (STEAM) 3-5 (STEAM) 6-8 9-12 College