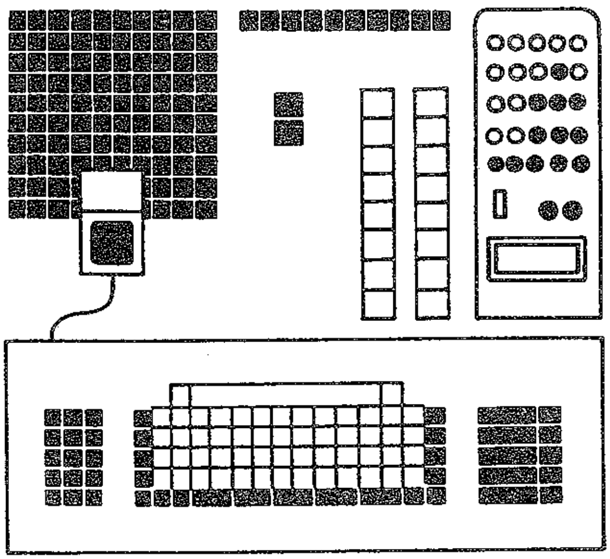
TEXAS MATHEMATICS TEACHER



A Simple Derivation of Trigonometric Identities

Solving Linear Diophantine Equations Using the Euclidean Algorithm and Convergents

Teaching Inference Schemes

MAY 1988

TEXAS MATHEMATICS TEACHER, the official journal of the Texas Council of Teachers of Mathematics, is published four times each year, in October, January, March, and May. All manuscripts and editorial correspondence should be addressed to George H. Willson, listed above.

Manuscripts should be neatly typewritten and double spaced, with wide margins, on 8 1/2 " by 11" paper. Authors should submit the signed original and four copies. Illustrations should be carefully prepared in black ink on separate sheets, the original without lettering and one copy with lettering added. As soon as possible after refereeing, authors will be notified of a publication decision. Originals of articles not accepted for publication will be returned to the authors. Two copies of the issue in which an author's manuscript appears will automatically be sent to the author.

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TEXAS MATHEMATICS TEACHER

TEXAS COUNCIL OF TEACHERS OF MATHEMATICS

Affiliated with the National Council of Teachers of Mathematics

May 1988

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President's Message

"Emptying is what makes a sponge worthwhile. It's not so bad to feel wrung-out if you know where to get filled up again."* Have you developed your filling up system so that you know where to get it quickly? Do you talk mathematics and how to best teach it with your colleagues? Do you take advantage of professional growth and Advanced Academic Training (AAT) courses offered in your district? Are you going to CAMT, the Conference for the Advancement of Mathematics Teaching, at the George R. Brown Convention Center in Houston, August 2-4?

If you've not already planned to be in Houston in August, do so now! CAMT starts even before its official beginning date of August 2. On Monday, August 1, there will be pre-conference workshops on geometry (getting ready for the 1990 Essential Elements) and FOM and Consumer Math (their new E.E.'s go into effect in 1989). Details about these workshops are in your CAMT program.

While you are thinking about going to CAMT, put on your agenda spending a few hours with me at the registration table and at the annual TCTM Breakfast. Last year we had Chamber of Commerce volunteers who worked many hours in registration. This year we'll have none of those. This year we expect more people and many more on-site registrations. The point here is that I'll need **lots** more help at registration. Look at your program and find a block of hours during which you can schedule working at the registration table at that time. Fill out the form on page five and send it to me by July 1.

Please note and tell EVERYONE that you must register for the **TCTM Breakfast** which will be at 7:00 a.m., Wednesday, August 3, at the convention center. Yes, it's free to members, but the caterer has to know, before the conference even begins, the number of breakfasts to prepare. We must pay for the number we order, so don't register until you're certain that you will attend. On page four you'll find the TCTM Breakfast Reservation Form. Return it to me and I'll mail your ticket to your HOME ADDRESS.

The absolute deadline for mailing your reservation is July 11.

Among the items on the agenda for the Breakfast are the introduction of Local Council Presidents and presentation of the map of new TCTM regions. The most significant event at the breakfast for me is at its conclusion when my term as President of TCTM ends. Being President of the Texas Council of Teachers of Mathematics for the last two years has been an extremely rewarding, filling experience for me. I've enjoyed working with the other officers and getting to know so many leaders in mathematics education across the State of Texas. I will miss this involvement, but I look forward to a very strong and active TCTM under the leadership of Otto Bieless, teacher and department chairman at Skyline High School in Dallas. Otto, I hope you enjoy being President of TCTM as much as I have!

Maggie Dement

*Bill Forbes, March 27, 1988

Note From the Editors

Please check the expiration date on your mailing label, your address and zip code. If any are incorrect, send your correction to John Huber, Sam Houston State University, P. O. Box 6768, Huntsville, Texas 77340. The data base for TCTM is kept by John.

When submitting journal articles, please send the signed original and four copies. Figures should be drawn on separate sheets and in black ink. The masthead page of the journal has complete directions.

TCTM BREAKFAST AND BUSINESS MEETING

It's become an annual event! This year at 7:00 a.m. on Wednesday, August 3, we will breakfast AT THE CONVENTION CENTER. See your CAMT program for the name of the room. The shuttle buses will run from the hotel at 6:50. If you are from the Houston area, you'll only have to park once, for the TCTM Breakfast and Business Meeting will be in the same place you'll spend the rest of the day! MAIL the reservation form below by July 11. This event is for members only and breakfast is BY RESERVATION ONLY. You will receive a ticket from me after I receive your reservation. The ticket will be taken at the door.

Mail by July 11 to: Maggie Dement
4622 Pine
Bellaire, Texas 77401

Name ______________________________

Home Address ______________________________

____

Home Telephone (____) ______________________________

Areas of Interest ________ K-5 ________ 6-8 ________ 9-12

Local Council Name ______________________________

Please reserve a place for me at the TCTM Breakfast and Business Meeting, Wednesday, August 3.

____
Signature

HELP NEEDED AT CAMT REGISTRATION

TCTM has the very important responsibility at the Conference for the Advancement of Mathematics Teaching (CAMT) of On-Site-Registration. During the entire conference the Registration Table will be open. Working there is exciting and fun! Every participant at CAMT must stop there, even if preregisterd. This conference is <u>for</u> teachers of mathematics, so <u>teachers</u> contribute some of your time to make it a success. The more volunteers we have, the fewer hours anyone will need to spend at the table. We expect more people than ever to attend CAMT this summer, and thus there will be more people than ever coming through registration. Tuesday, August 2, will be the heaviest traffic time and therefore when I need the most help. Wednesday and Thursday are also important, for someone must be at the table at all times.

Use this form to volunteer to work at the registration table.

Mail it to: Maggie Dement
4622 Pine
Bellaire, Texas 77401

I will work at the Registration Desk during CAMT, August 2-4, 1988 at the George R. Brown Convention Center in Houston, Texas.

Times I will help ______________________________

Times I can <u>not</u> help ______________________________

Times don't matter, assign me whenever you most need me _______

Name ______________________________

Home Address ______________________________

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A Simple Derivation of Trigonometric Identities

Bella Weiner

Pan Am University
Edinburgh, Texas

Trigonometric identities occupy an important place in trigonometry and precalculus courses and are frequently used in the integration of trigonometric expressions. Students often complain there are too many trigonometric identities to memorize because they fail to recognize different forms of a small number of identities. To overcome this difficulty, it should be repeatedly emphasized that trigonometric identities obey the same algebraic rules as all other equations and may be solved for any term depending on the requirements of the problem. Furthermore, it is useful to derive the major trigonometric identities directly from the definitions of the trigonometric functions rather than develop some of them from the others. This helps the students better understand they can work with trigonometric functions in the same manner as with all other functions studied earlier in the algebra courses. Certainly, new functions have new properties, and the properties of trigonometric functions constitute the central part of trigonometry. At the same time, we should also take great care to alleviate the students difficulties in performing algebraic operations with trigonometric functions and expressions.

The introduction to trigonometric identities starts with repeating the definitions of trigonometric functions

$$\sin\theta = \frac{y}{r}, \cos\theta = \frac{x}{r}, \tan\theta = \frac{y}{x}, \cot\theta = \frac{x}{y},$$
$$\sec\theta = \frac{r}{x}, \csc\theta = \frac{r}{y},$$

where (x,y) is any point other than the origin on the terminal side of the angle θ in standard position, and r is the distance from (x,y) to the origin. We also need to remind and briefly review some concepts from arithmetic and algebra. We say that two numbers or expressions are reciprocals if their product is 1. The numbers 2 and 1/2 are reciprocals since 2 (1/2) = 1, the same is truce for - 3/2 and -2/ 3, or $\sqrt{2}$ and $\sqrt{2}/2$. Likewise, x and 1/x are reciprocals for all numbers x, except x = 0, because x (1/x) = 1. Then, students are reminded about identities, statements such as $2x = x + x$, $x^3 = x \cdot x \cdot x$, $(x - 4)(x + 4) = x^2 - 16$, which they remember from algebra. These equations are identities because they are true for all replacements of the variable for which they are defined. Equations that involve the trigonometric functions and are valid for all possible values of the independent variable are called trigonometric identities. Showing that a trigonometric equation is an identity is referred to as verifying the identity. It is often necessary to simplify complicated expressions involving various trigonometric functions. Therefore, trigonometric identities are of special importance among all identities, and we emphasize in class that being able to prove trigonometric identities is a very useful skill developing of which, however, takes a lot of practice.

1. Reciprocal Identities. The basic trigonometric identities are all derived directly from the definitions of the trigonometric functions. In many cases, complicated trigonometric expressions can be simplified by transforming the functions involved in terms of the sine and cosine alone. Naturally, a question students can easily answer is: What pairs of trigonometric functions in the above definitions are reciprocals? The definitions indicate that the sine and cosecant, cosine and secant, tangent and cotangent functions are reciprocals because

$$\sin\theta \csc\theta = \frac{y}{r} \cdot \frac{r}{y} = 1, \quad \cos\theta \sec\theta = \frac{x}{r} \cdot \frac{r}{x} = 1,$$
$$\tan\theta \cot\theta = \frac{y}{x} \cdot \frac{x}{y} = 1.$$

From here, we get three basic identities

$$\csc\theta = \frac{1}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta}, \cot\theta = \frac{1}{\tan\theta}. \qquad (1)$$

Of course, students should be reminded that any trigonometric identity can be changed, by appropriate algebraic operations, to equivalent forms depending on the requirements of different problems. For instance, we can also write

$$\sin\theta = \frac{1}{\csc\theta}, \cos\theta = \frac{1}{\sec\theta}, \tan\theta = \frac{1}{\cot\theta},$$

but the basic identities are given by formulas (1).

2. Ratio Identities. There are two ratio identities that express tangent and cotangent as functions of sine and cosine. Given proper directions, students will readily obtain them. Taking $\tan\theta = y/x$ and dividing both numerator x and denominator y by r yields

$$\tan\theta = \frac{y}{x} = \frac{y/r}{x/r} = \frac{\sin\theta}{\cos\theta}.$$

The same operation applied to the cotangent function gives

$$\cot\theta = \frac{x}{y} = \frac{x/r}{y/r} = \frac{\cos\theta}{\sin\theta}.$$

These two relations also belong to the basic trigonometric identities.

3. Pythagorean Identities. The derivation of the reciprocal and ratio identities is fairly easy for students because it is based only on the operation of division. However, the relationship between the sine and cosine functions relies on the Pythagorean theorem

$$x^2 + y^2 = r^2 \qquad (2)$$

and therefore cannot be obtained by the students independently. They should be advised to divide (2) term-by-term by r^2,

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1,$$

then to recant the latter equation as

$$(x/r)^2 + (y/r)^2 = 1$$

and to compare the left-hand side with the definitions of $\cos\theta$ and $\sin\theta$. Only upon these directions, students discover the major trigonometric identity

$$\cos^2\theta + \sin^2\theta = 1. \tag{3}$$

After this, we discuss in class different forms of this basic identity and urge the students to divide (2) term-by-term by x^2, which yields

$$1 + (y/x)^2 = (r/x)^2$$

that is, the basic identity

$$1 + \tan^2\theta = \sec^2\theta. \tag{4}$$

Finally, dividing (2) term-by-term by y^2, students get the basic identity

$$\cot^2\theta + 1 = \csc^2\theta. \tag{5}$$

This derivation of the last two identities is somewhat different from that given in the textbooks, where (4) and (5) are obtained by successively dividing (3) by $\cos^2\theta$ and $\sin^2\theta$ (see 1, 2, 3).

Before being in a position to verify new, more complicated identities, students should memorize the basic identities. This significantly improves the chances of choosing better options and effective strategies, since there are no standard steps to take to simplify trigonometric expressions. A few general guidelines that can be of help proving trigonometric identities should be stated before solving examples. Namely, it is often best to proceed by first writing down the more complicated side of the identity and gradually transforming it to the simpler one. This transformation will come about by using the basic identities and the rules of algebra. One useful technique is to rewrite the more complicated side in terms of sines and cosines only. Simplifying trigonometric expressions and verifying trigonometric identities penetrates the entire trigonometry course. And there can be no substitute for experience in this area, for only by trial and error, by solving a sufficient number of examples students learn what will work in which situations.

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Kelly, T. J., Balomenos, R. H., & Anderson, J. T. (1987). Trigonometry, Houghton Mifflin.

McKeague, C. P. (1984). Trigonometry, Academic Press.

Roman, S. (1987). Precalculus, Harcourt Brace Jovanovich.

REFEREES WANTED

Manuscripts published in the Texas Mathematics Teacher are reviewed by at least three mathematics educators. Members of TCTM who are regular readers of the journal - classroom teachers, supervisors, and teacher educators - who would like to review manuscripts should write to the TCTM indicating their willingness to serve and the level of interest (Elementary, Secondary, or both). Contact George H. Willson, Box 13857, University of North Texas, Denton, Texas 76203-3857. The Editorial Panel will review the responses and make the final selection.

Solving Linear Diophantine Equations Using the Euclidean Algorithm and Convergents

Betty Travis

The University of Texas at San Antonio
San Antonio, Texas

Topics from Number Theory can be used to provide challenging classroom problems for students, yet many of these topics do not require an extensive background in mathematics. For example, many students are familiar with the use of the Euclidean Algorithm to find the greatest common divisor (gcd) of two integers, yet few may be aware of some other uses of the procedure. This paper examines the use of the algorithm to solve linear Diophantine equations, making particular use of the "partial" quotients that are derived in the use of the algorithm.

For example: Solve the linear Diophantine Equation

$$3125x + 1024y = 1.$$

(A linear Diophantine equation, named after the early mathematician Diophantus, is a linear equation in two or more variables whose solution must be <u>integers</u>.)

A frequently used method to solve this given problem involves a "forward" approach of using the Euclidean algorithm to express the gcd of two integers as a linear combination of those integers. For example, this problem would be solved as follows:

$$33125 = 1024(3) + 53 \qquad (1)$$
$$1024 = 53(19) + 17 \qquad (2)$$
$$53 = 17(3) + 2 \qquad (3)$$
$$17 = 2(8) + 1 \qquad (4)$$
$$2 = 1(2) + 0$$

We solve for the remainders in (1) - (4).

$$53 = 3125 - 3(1024) \quad (5)$$
$$17 = 1024 - 19(53) \quad (6)$$
$$2 = 53 - 3(17) \quad (7)$$
$$1 = 17 - 8(2) \quad (8)$$

We substitute (5) into (6):

$$17 = 1024 - 19[3125 - 3(1024)] = 58(1024) - 19(3125) \quad (9)$$

Using this result and (5) into (7), we have:

$$2 = [3125 - 3(1024)] - 3[58(1024) - 19(3125)]$$

$$= 58(3125) - 177(1024).$$

Substituting this last expression and (9) in (8) gives:

$$1 = [58(1024) - 19(3125) - 8[58(3125) - 177(1024)]$$

$$= 1474(1024) - 483(3125)$$

Therefore (x,y) = (-482,1474).

But another approach of solving $ax + by = k$, where gcd (a,b) = k, is to use convergents and continued fractions. (The term "convergent," a commonly used term in Number Theory, derives from the expansion of a rational number into a continued fraction. See Continued Fractions, published as enrichment material for high school students by the Mathematical Association of America, for a complete treatment of the topic of convergents and continued fractions.)

We begin with

$$3125x + 1024y = 1$$

and the Euclidean Algorithm to find the partial quotients (3, 19, 3, 8, 2).

```
1024)3125(3
     3072
       53)1024(19
           1007
             17)53(3
                51
                 2)17(8
                   16
                    1)2(2
                      2
```

We now build a convergent table, using the successive partial quotients obtained above. We will call these partial quotients d_i; hence $d_1 = 3$, $d_2 = 19$, $d_3 = 3$, $d_4 = 8$, and $d_5 = 2$. The algorithm we now expose asserts that each convergent of the table can be written as a fraction:

$$c_i = \frac{p_i}{q_i}$$

We begin by assigning $p_{-1} = 0$, $p_0 = 1$, $q_{-1} = 1$ and $q_0 = 0$. (This assignment is made so that all values of c_i are valid.) The algorithm now defines p_i and q_i:

$$c_i = \frac{p_i}{q_i}, \quad \text{where } p_i = d_i p_{i-1} + p_{i-2}, \quad \text{where } q_i = d_i q_{i-1} + q_{i-2}$$

Using the just stated values for p_{-1}, p_0, q_{-1} and q_0 as well as the partial quotients d_i and the algorithm just defined, we can calculate each value of c_i.

$$(i\text{-}1) \quad c_1 = \frac{p_1}{q_1} = \frac{d_1 p_0 + p_{-1}}{d_1 q_0 + q_{-1}} = \frac{3(1) + 0}{3(0) + 1} = \frac{3}{1}$$

$$(i\text{-}2) \quad c_2 = \frac{p_2}{q_2} = \frac{d_2 p_1 + p_0}{d_2 q_1 + q_0} = \frac{19(3) + 1}{19(1) + 0} = \frac{58}{19}$$

$$(i\text{-}3) \quad c_3 = \frac{p_3}{q_3} = \frac{d_3 p_2 + p_1}{d_3 q_2 + q_1} = \frac{3(58) + 3}{3(19) + 1} = \frac{177}{58}$$

$$(i\text{-}4) \quad c_4 = \frac{p_4}{q_4} = \frac{d_4 p_3 + p_2}{d_4 q_3 + q_2} = \frac{8(177) + 58}{8(58) + 19} = \frac{1474}{483}$$

$$(i\text{-}5) \quad c_5 = \frac{p_5}{q_5} = \frac{d_5 p_4 + p_3}{d_5 q_4 + q_3} = \frac{2(1474) + 177}{2(483) + 58} = \frac{3125}{1024}$$

We now write our table as follows:

i	1	2	3	4	5
d_i	3	19	3	8	2
c_i	3/1	58/19	177/58	1474/483	3125/1024

The values of c_i are the convergents and they "converge" to a fraction that will always be the coefficients of the linear Diophantine equation (i.e., 3125 and 1024). Theorems from Elementary Number Theory state that the last convergent listed is always a/b (from $ax + by = 1$) in lowest terms and the difference between any two successive pairs of cross-products is always equal to one. (For example, use the fractions 58/19 and 177/58. The cross-products are (58)(58) = 3364 and (19)(177) = 3363, numbers whose difference is one.) These theorems allow us to solve the given equation (3125x + 1024y = 1) by inspection since we were looking for products whose difference was equal to one.

Be examining the original problem

$$3125x + 1024y = 1$$

and the table values, we see that $|x| = 483$ and $|y| = 1474$.

Since $3125(483) = 1509375$ and $1024(1474) = 1509376$, then $x = -483$ so that the difference of these two products is positive. Hence, $(x,y) = (-483, 1474)$ will be a solution to the original Diophantine equation.

This approach of solving a linear Diophantine Equation can be an interesting problem to investigate with advanced high school mathematics students since it introduces several basic principles of number theory, including convergents and continued fractions, which generally do not appear elsewhere in the curriculum. Also, because of the recursive nature of the algorithm, it becomes an excellent exercise for students in Computer Mathematics.

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Teaching Inference Schemes

Kenneth A. Retzer

Illinois State University
Normal, Illinois

Proof making needs to receive consideration as a special (but common) case of nonroutine problem solving. In the effort to make problem solving a focus of school mathematics, nonroutine problems have been devised and infused throughout the mathematics curriculum. But more attention needs to be given to the required proof making already present in the curriculum.

Traditional problem solving models can be helpful in describing stages of proof making activities. Recall that a typical proof in plane geometry or algebra is a sequence of statements, each statement often matched with a supporting reason drawn from mathematics content. Making a plan and carrying out that plan in proof making involves selecting (or composing) those statements and putting them in a "logical" sequence. The sentences are sequenced correctly provided that each new conclusion follows logically from some of the preceding sentences. Each of these subconclusions "follow logically" because each can be justified by an underlying valid inference scheme. It has been argued elsewhere (Retzer, 1984a, 1985) that explicitly teaching some valid inference schemes in plane geometry may be a valuable asset in student proof making activities. A minimal list of inference schemes to be taught has been suggested (Retzer, 1984b) and how they could be used in writing selected proofs was described (Retzer, 1984c).

Truth tables get their fair share of attention in the logical aspects of the mathematics curriculum. But students need to know some of the most useful inference schemes and need to be convinced that they are valid i.e., produce true conclusions.

Recall that each inference scheme has a set of sentences, called premises, and a single sentence, called a conclusion. If an inference scheme is valid, it is impossible for the premises to be true and the conclusion false. Given that understanding of validity, it is impossible to form a valid inference scheme with a set of true premises for which the desired conclusion could be false in some cases and true in others.

Other than expressing the need in the mathematics curriculum described above and refreshing our memories on the concepts involved, the purpose of this article is to discuss the teaching of some inference schemes and, in particular, describe a common but seldom used method of using truth tables to convince students that inference schemes are valid. It is a clearer and simpler method than connecting the two using tautology and equivalent statements. This method (Retzer, in press) also allows them to discover a correct obvious conclusion without prior knowledge of some inference schemes. They can also see why some sets of premises cannot be used to form a valid inference scheme since it cannot yield a unique true conclusion.

VALIDATING THE MOST FAMOUS INFERENCE SCHEME

Probably the most famous inference scheme is called Modus ponendo ponens (PP) by Suppes (Suppes, 1968). Other texts call it the law of detachment, and middle age scholars called it the Bridge of Asses because it was considered so fundamental that even a burro could reason well enough to draw the conclusion it calls for. For example, we tell Aggie seniors, "If you graduate as an Aggie, then you will go places." Believing that to be true, he/she graduates - and waits expectantly by the highway.

We can use a truth table to show the validity of the inference scheme PP. Let us assume that the student knows and understands the truth table for implication (P-->Q).

TABLE 1

	P	Q	P --> Q
1)	T	T	T
2)	T	F	F
3)	F	T	T
4)	F	F	T

We want to show the validity of the inference scheme Modus ponendo ponens (PP). It is usually written:

P --> Q
P
Q

This means that both P-->Q and P are assumed true. We use these two premises and the process of elimination to establish that Q is true since it cannot be false given the truth of the premises.

Since P-->Q is assumed true, each row in the truth table in which P-->Q is false must be eliminated because it does not meet the condition. Thus, row 2 of the truth table is eliminated.

TABLE 1 - A

	P	Q	P --> Q
1)	T	T	T
3)	F	T	T
4)	F	F	T

P, if used as a premise, must be assumed true. Since P is false in rows 3 and 4, both rows can be deleted.

TABLE 1 - B

	P	Q	P -> Q
1)	T	T	T

Therefore, row 1 is the only row in which both premises hold. An examination of row 1 leads to the conclusion that Q is true since that is the only truth value for Q which is left.

We need to believe the validity of Modus ponendo ponens for it is the workhorse for many of our proofs of congruence of triangles. We let some conjunction of conditions, such as SAS, serve the role of the P in the general inference scheme and the definition of congruence of form P<-->Q to get the other premise P-->Q. The conclusion of congruent triangles, in the form Q, follows, and students are capable of understanding and benefitting from explicit discussion of the underlying logic.

DISCOVERING A TRUE CONCLUSION

PP is usually easy for students to see even without the use of truth tables. It is not as easy for the average student to discover a true conclusion with *Modus tollendo tollens* (TT) (continuing to use Suppes' names for inference schemes). What conclusion, a student may be asked, can be drawn if P-->Q is true and Q is false, (i.e., ~Q is true)? Referring to the same truth table in TABLE 1, since P-->Q is assumed true, row 2 can be eliminated.

The assumption that Q is false eliminates row 1 and 3.

TABLE 1 - C

	P	Q	P --> Q
1)	T	T	T
3)	F	T	T
4)	F	F	T

TABLE 1 - D

	P	Q	P --> Q
4)	F	F	T

Since row 4 is the only row left, the student will be able to see that the only possible conclusion for P is that it is false. Thus TT is validated and may be symbolized:

$$\begin{array}{l} P \rightarrow Q \\ \underline{\sim Q \quad} \\ \sim P \end{array}$$

Modus tollendo tollens (TT) could be discovered by a hapless athlete in the following manner. The school policy affirms, "If you have high enough grades, you can play football." The student is denied the right to play, so he concludes that he did not have sufficiently high grades.

DISCOVERING WHY A TRUE CONCLUSION DOES NOT EXIST

After a few examples such as those above, students will be able to use truth tables to help them to establish true conclusions, but, in addition, they will be able to see why some premises will not yield a true conclusion. For example, we ask them, can we draw a true conclusion about Q if we know that P-->Q is true and P is false (i.e., ~P is true)?

That P is false (i.e., ~P is the desired true premise) eliminates both rows 1 and 2 from our original TABLE 1. That P-->Q is assumed true independently eliminates row 2.

TABLE 1 - E

	P	Q	P --> Q
3)	F	T	T
4)	F	F	T

This leaves rows 3 and 4 as possibles. The student will be able to see that Q is true in row 3 and false in row 4. Since Q can be either true or false, no valid conclusion can be drawn with the above premises. In symbolic form, both of the following schemes are invalid.

P --> Q	P --> Q
-P	-P
-Q	Q

For example, a person can reasonably believe that, "If Notre Dame loses the Cotton Bowl, the Aggies will be happy." However, even if we are sure Notre Dame lost, we cannot be sure about the mental state of the Aggies. Rumor has it that a variety of things affect Aggies.

To use this method to convince a student that Modus tollendo tollens is valid and that the two forms of the inference scheme above are not valid can provide the basis to discuss the two most common misuses of Modus ponendo ponens, the most important inference scheme underlying proofs in high school plane geometry.

VALIDATING *MODUS TOLLENDO PONENS* (TP)

Validity of inference schemes other than those involving implication are just as easy using truth tables. Consider

TABLE 2

	P	Q	P ∨ Q
1)	T	T	T
2)	T	F	T
3)	F	T	T
4)	F	F	F

What happens if we take P ∨ Q and ~P as our premises? That P ∨ Q is assumed true eliminates row 4.

TABLE 2 - A

	P	Q	P ∨ Q
1)	T	T	T
2)	T	F	T
3)	F	T	T

~P used as a premise means P is false. That P is false eliminates rows 1 and 2 leaving:

TABLE 2 - B

	P	Q	P ∨ Q
3)	F	T	T

This leaves us with just row 3 and we can draw the conclusion that Q is true. We have thus established the validity of Modus tollendo ponens (TP), which can be written in symbolic form:

$$\frac{\begin{array}{c} P \vee Q \\ -P \end{array}}{Q}$$

To illustrate a use of Modus tollendo ponens (TP), one might consider a good old boy who shocks his wife with, "I bought myself a new red pickup truck or I bought myself a new red dress today." Imagine her relief if he further assures her, "I did not buy myself a red dress," for she can decide what to conclude.

Modus tollendo ponens is commonly used in a strategy called *proof by elimination* which is helpful in a variety of proofs in algebra and analysis.

ESTABLISHING HYPOTHETICAL SYLLOGISM

In the above examples, the only time we were able to draw an obvious true conclusion was when all but one row was eliminated. Moving to a more complex example, we see that this is not always the case as we establish the validity of Hypothetical syllogism (HS):

$$\frac{\begin{array}{c} P \rightarrow Q \\ Q \rightarrow R \end{array}}{P \rightarrow R}$$

by use of TABLE 3.

TABLE 3

	P	Q	R	P --> Q	Q --> R	P --> R
1)	T	T	T	T	T	T
2)	T	T	F	T	F	F
3)	T	F	T	F	T	T
4)	T	F	F	F	T	F
5)	F	T	T	T	T	T
6)	F	T	F	T	F	T
7)	F	F	T	T	T	T
8)	F	F	F	T	T	T

That P-->Q is true eliminates rows 3 and 4.

TABLE 3 - A

	P	Q	R	P --> Q	Q --> R	P --> R
1)	T	T	T	T	T	T
2)	T	T	F	T	F	F
5)	F	T	T.	T	T	T
6)	F	T	F	T	F	T
7)	F	F	T	T	T	T
8)	F	F	F	T	T	T

That Q-->R is true eliminates rows 2 and 6.

TABLE 3 - B

	P	Q	R	P --> Q	Q --> R	P --> R
1)	T	T	T	T	T	T
5)	F	T	T	T	T	T
7)	F	F	T	T	T	T
8)	F	F	F	T	T	T

This means that we must consider rows 1, 5, 7, and 8 as supporting the validity of the inference scheme under the given conditions. But in all those rows P-->R is true. Therefore, we can conclude P-->R is always true under the given conditions, i.e., when P-->Q and Q-->R are true.

To know that Hypothetical syllogism (HS) is a valid inference scheme would permit one to accept the statements, "If you earn more, you will pay more taxes," and "If you pay more taxes, then you will help the government support unpopular causes," and correctly conclude, "If you earn more, then you will help the government support unpopular causes." Unfortunately, if we stop earning, we can become supported by one of these unpopular causes - public welfare.

Hypothetical syllogism is used most often in longer proofs to connect a chain of implications and permit modus ponendo ponens to be used with the first antecedent and the last consequent in that chain.

After just a few examples, this truth table method of validation of inference schemes can be used independently by the student. The student can determine from a partial inference scheme whether or not an obvious true conclusion can be drawn. Furthermore, the concept of inference schemes can be taught as a plausible companion of the more familiar truth tables.

From the perspective of Polya's general problem solving model, understanding the given and the "to prove" are the specific counterparts of understanding a nonroutine problem. Understanding underlying inference schemes can help students put the statements they creatively select into logical sequences and, thus, can aid in carrying out the proof making plan. Finally, a knowledge of inference schemes is indispensable if we are going to enable students, as well as teachers, to determine the validity of the reasoning used in proofs. Only with knowledge of inference schemes is the look back stage for proof making possible.

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TCTM JOURNAL NEEDS ARTICLES
FOR ALL LEVELS OF MATHEMATICS

STUFF

*****Strategic Tactics Ultimately For Fun*****

Beverly Millican

Elementary Mathematics Coordinator
Plano Independent School District

In the March, 1988 issue of Texas Mathematics Teacher, several elementary activities for instructing place value in a game format at the concrete level were presented. No paper and pencil were required for these activities, as the emphasis was totally on manipulating the materials and making the appropriate exchanges. This article will focus primarily on instructing place value with games at the "connecting level."

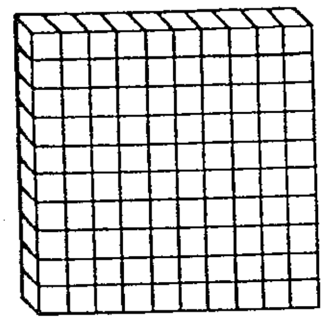
While it is not a requirement that students begin with the previous concrete activities, it is highly recommended that students be given ample time to work with concretes before "bridging" to the abstract level. Several activities will be presented here that promote childrens' functioning at this "connecting level," in that students will simultaneously use concrete materials and record their actions with them symbolically. In a developmental mathematics sequence, this stage is critical for linking the meaning exemplified by the concrete materials with their abstract representatives.

As in the previous issue, a description and adaptation of a place value game included by Marilyn Burns in her book The Math Solution will be described. Materials required are Base 10 blocks:

"units"

"longs"

and "flats"



The gameboard can be made for each player by using a legal size piece of paper, divided into three columns and labeled at the top, as shown.

F	L	U

This "FLU" board (for Flats, Longs and Units) board helps students organize their materials, facilitates the exchanges that have to be made when regrouping, and helps provide the connection with numbers that must be made when recording at the symbolic level.

WHO HAS MORE

MATERIALS: FLU boards
Base 10 materials
1 die or number cube
1 Recording Sheet

Who has more?	
Kate	John

WHO HAS MORE? Two players take turns tossing the die. The first toss tells how many flats the first player places on his FLU board. (His partner then tosses the die and places the appropriate amount of flats on his own board.) The second toss tells how many longs for the first player to place on his board. (Partner repeats.) The third toss tells how many units need to be placed on his board. (Partner repeats.) Next, on the recording sheet, each player writes in his column the number he has just built on his board. They circle the one which is more.

Who has more?	
Kate	John
201	(642)

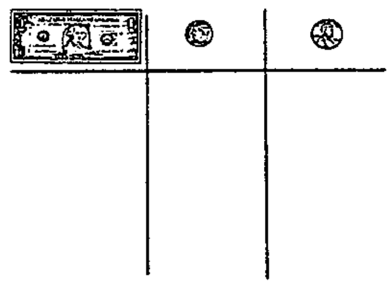
It is interesting to note teachers' reaction to this game. Several have commented that "the game is over after each player has made his first roll." Of course this is true, but young children are not always so quick to recognize this. After all, isn't that the point of the game in instructing the meaning of place value?

VARIATION: You may want to increase the "suspense" with older children by adding a MORE/LESS spinner with the game. Retitle the game to "Who Has More or Less?" Now in order to determine who wins, after the recordings have been made on the chart, the first player spins the spinner. If the spinner lands on MORE, the player with the larger number circles the number and "wins." If LESS is spun, then the player with the smaller number circles and "wins."

Penny McAdoo at Region 10 ESC has created a "connecting level" place value game to be played with money (pennies, dimes, and dollars) instead of Base 10 materials. While Base 10 blocks are a clear, proportional model, the money model has the strength of being directly connected to the real world.

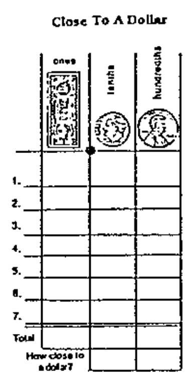
The gameboard is similar to the FLU board in that it has three columns, labeled with pennies, dimes, and dollars.

Although imitation "dollars are used, real pennies and dimes, supplied by the teacher, or even better, brought by students from home, add excitement, but demand firm management.



One baggie per pair of students is required and needs to contain:

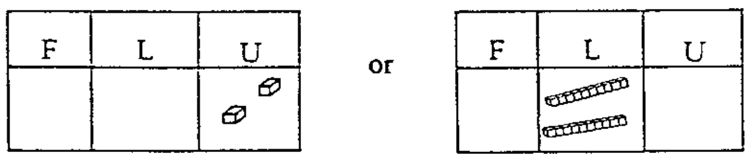
25 pennies
20 dimes
1 dollar bill
1 die



CLOSE TO A DOLLAR

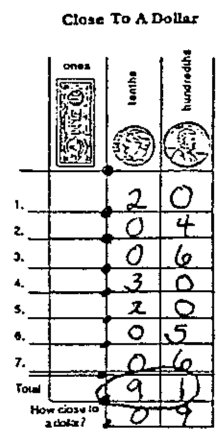
MATERIALS: Money gameboard
"Close to a Dollar" recording sheet for each player
1 baggie of money and dice per pair of players

CLOSE TO A DOLLAR: The object of this game is to see which player can come the closest to one dollar (exceeding the one dollar amount is permitted). Using only 1 die, players alternate rolling the die. Both players use the number that appears on the first roll and independently decide whether to take that amount of pennies or dimes from the "bank" and place on their gameboards. For instance, if a "2" is thrown, I may choose to build



according to my strategy. This procedure is followed until all 7 rolls have been made. Each player must play all 7 times.

To add the "connecting level" to this game, introduce the "Close to a Dollar" recording sheet as a record of what the children build on their gameboards with each throw. Encourage children to continue building with the materials on their gameboards so that the chart is only a record of their actions and not the focus of the activity. Remember that children need to relinquish the concrete materials only when they are ready, and not when we as teachers think they should.



Look for 2 more place value games to be described in the fall section of STUFF. Be ready for the quantum leap to the abstract level in this developmental mathematics instruction of place value. Stay tuned....

SCHOLARSHIPS FOR FIRST-TIME CAMT GOERS

Are you a TCTM member who has never attended the Conference for the Advancement of Mathematics Teaching? Is your trip to CAMT this summer not being funded by your district or local math council? Do you have a position for the fall of 1988 in which you teach math or influence math education? If your answer to these three questions was "yes," then you are eligible to apply for one of the two TCTM $150 scholarships for CAMT. Winners will be notified by the end of June and will be introduced and checks presented at the TCTM Breakfast, August 3. Be sure to register for the Breakfast.

On this form get the appropriate signature and mail it with the recommendation and your statement by June 10 to:

Maggie Dement
4622 Pine
Bellaire, Texas 77401

Name ____________________
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Home Telephone (   ) ____________________
Local Math Council ____________________
Position (Be Specific) ____________________

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   b) Is not receiving district or math council funds for the trip this August 2-4.
   c) Has a position teaching mathematics or influencing math education for the fall of 1988.

____
Signed by Principal or Supervisor

2) Include a brief recommendation from your principal or supervisor.
3) Include an account by you of how the ideas you get at CAMT will be shared with colleagues and how those ideas may influence your job performance.

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George H. Willson, Editor
Texas Council of
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P. O. Box 13857
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