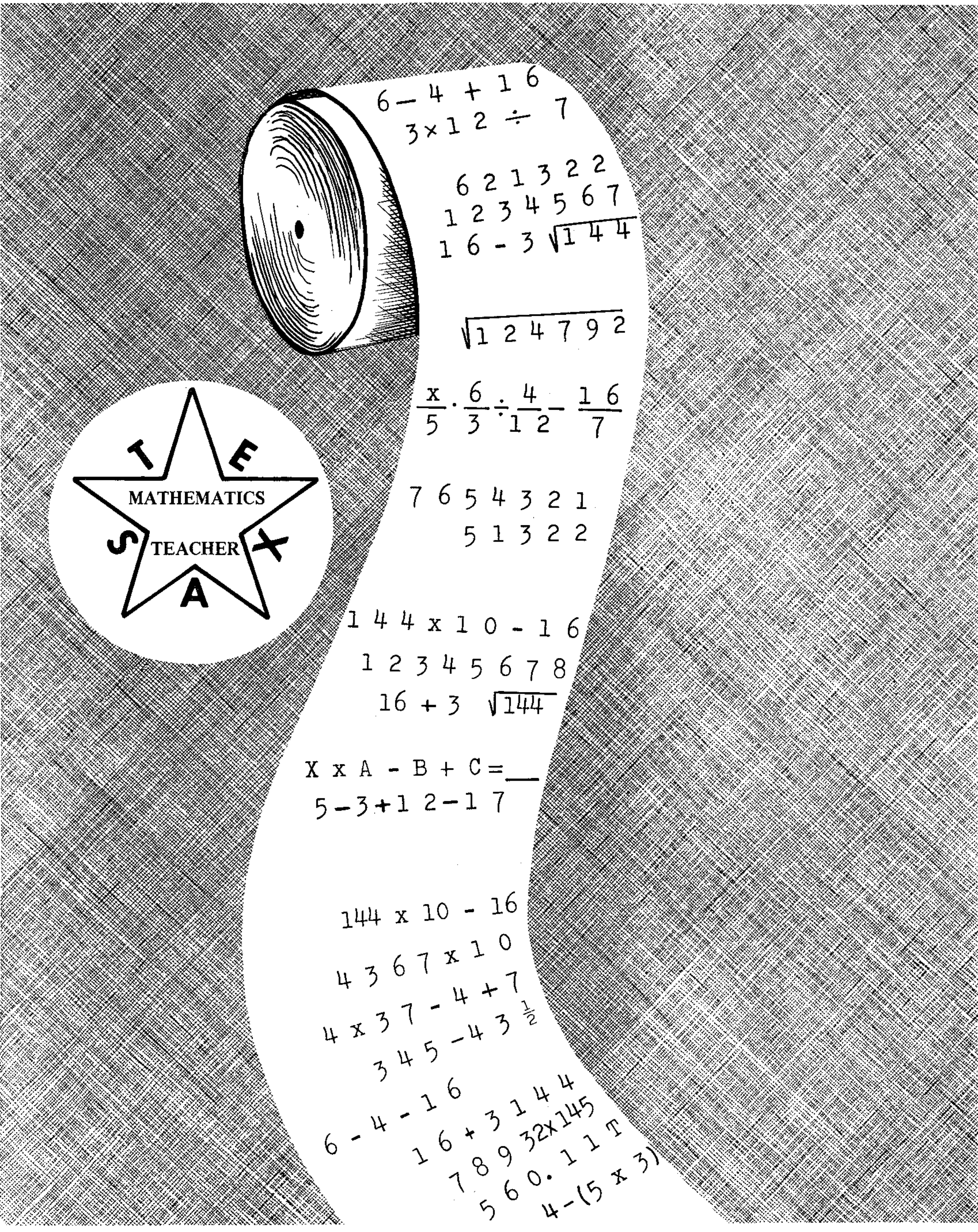


TEXAS
MATHEMATICS
TEACHER
6 − 4 + 1 6
3 × 1 2 ÷ 7
6 2 1 3 2 2
1 2 3 4 5 6 7
1 6 - 3 √1 4 4
√1 2 4 7 9 2
x/5 · 6/3 ÷ 4/12 − 16/7
7 6 5 4 3 2 1
5 1 3 2 2
1 4 4 x 1 0 - 1 6
1 2 3 4 5 6 7 8
16 + 3 √144
X x A - B + C = ___
5 − 3 + 1 2 − 1 7
144 x 10 - 16
4 3 6 7 x 1 0
4 x 3 7 - 4 + 7
3 4 5 − 4 3 ½
6 - 4 - 1 6
1 6 + 3 1 4 4
7 8 9 32x145
5 6 0. 1 1 T
4-(5 x 3)

TEXAS MATHEMATICS TEACHER is the official journal of the Texas Council of Teachers of Mathematics. The views expressed are the contributor's own and are not necessarily those of the publisher or the editor. All manuscripts and correspondence about this publication should be addressed to Mr. J. William Brown, *Texas Mathematics Teacher,* 100 So. Glasgow Drive, Dallas, Texas 75214.

TEXAS MATHEMATICS TEACHER is published quarterly by the Texas Council of Teachers of Mathematics. Payment of membership fee of $5.00 entitles members to all regular Council Publications.

TEXAS MATHEMATICS TEACHER

Texas Council of Teachers of Mathematics

Vol. XXXI January, 1984 No. 1

PRESIDENT'S MESSAGE

Dear TCTM Members,

Happy Holidays! The new year is the perfect time for resolutions and new beginnings. I hope all of you included in your list of resolutions the commitment to get involved with your professional organizations. TCTM needs your help and support!

Included in this issue of the journal is a copy of the position paper passed at the TCTM meeting at CAMT. Paul Forester and Ray Carry worked diligently to get this revised statement into the present form. We will be sending this forward to appropriate state and national committees. I urge you to read the statement and endorse our position of helping state groups to understand certain aspects of the problems facing mathematics education in our state. Contact your regional representative if you have any comments and suggestions.

Best wishes for a joyous New Year.

Sincerely,

Betty Travis

A POSITION STATEMENT ON TEACHER EVALUATION BY THE TEXAS COUNCIL OF TEACHERS OF MATHEMATICS

The National Commission on Excellence in Education presented its report to the Secretary of Education in April 1983. In this report, A Nation At Risk, one recommendation calls for an effective evaluation system that includes peer review so that superior teachers can be rewarded, average ones encouraged, and poor ones either improved or terminated.

It has been suggested by some that it is enough to compute the mean grade equivalent gains on standardized tests made by a teacher's students to determine the teacher's effectiveness. This position is relatively easy to apply but it is subject to serious logical faults.

An evaluation of teacher effectiveness based on the mean gains measured by standardized tests would result in invalid and unreliable rankings of teachers. It would be invalid because the proportion of items covering individual topics on standardized amthematics tests is not representative of the instructional time spent on these topics. It would also be invalid because the range of content sampled by such tests is much more restricted than the range of content included in textbooks and that are required by good curriculum guides. Such a system would be unreliable because gains on standardized tests are correlated with such variables as student intelligence, student prior achievement, textbooks used and socio-economic level — all of which are not subject to modification by improved teaching.

Another error inherent in using standardized test scores as the only criterion for evaluating teachers is the statistical phenomenon of regression to the mean. Because of errors in measurement, classroom groups measured as below average in the fall are more likely to show higher gains in the spring than the true gains in actual knowledge. Similarly the measured gains for classroom groups who score above average in the fall are more likely to be less than the actual gains in knowledge. Thus the odds favor greater gains for groups that score low in the fall testing.

The use of standardized test scores as the sole criterion essentially renders the school administrator impotent in the evaluation process by reducing the evaluation to unreliable numbers that have an unjustified appearance of accuracy.

Conclusion:

In view of the above the Texas Council of Teachers of Mathematics takes the following position:

1. Evaluation of Mathematics Teachers should not be based upon student performance on standardized tests.
2. Criteria for evaluation of Mathematics Teachers should include:
   a. Structured review by school district personnel with respect to clearly defined goals.
   b. Peer review as recommended by the National Commission on Excellence in Education.
3. Additional criteria should include:
   a. A statement of self evaluation.
   b. Student evaluations.

MATHEMATICS IS FUN?

by Marsha Rosson
Texas Tech University

The reaction of most students when mathematics is mentioned is usually "Ughi", "I hate mathi", or "I do not understand math!". How am I as a mathematics teacher to alleviate this negativism? Mathematics can be fun and this article is an attempt to prove just that.

The most obvious way to have fun with mathematics is playing games. Liedtke, (1980) states that, "games and game settings lend variety to teaching methods or class-room settings, and they can provide an added purpose to the review of learned concepts, ideas, and skills. In a game setting, young children have an opportunity to collaborate and interact socially with other children, an experience that is valuable in terms of their intellectual development." Games do add some variety to otherwise "dull and routine" mathematics lessons. Mathematics can be fun in other ways also.

Zoltan Dienes advocates mathematical play. He says that mathematical play can be generated simply by providing children with a large variety of constructed mathematical materials. He also advocates unstructured play prior to structured play, which involves actually learning through a game situation using constructed mathematical materials.

Robert Wirtz says that children already understand the concepts of addition, subtraction, multiplication, and division when they enter school. The problem lies with the symbols I use to represent these concepts. Fun can be incorporated into learning these concepts by using manipulative objects such as beans, candy, posicle sticks, and Cuisenaire Rods.

Henry Lunde, (1980) taught several mathematical concepts to his nine and ten year old students with M & M's. He gave each student one package of M & M's and then asked them to guess the amount of M & M's in the bag. They were then told to open the bags and count the M & M's. Their next task was to separate the candies by color. They were to use the signs $>$, $<$, or = to show relationships between the different color groups of M & M's. Addition, subtraction, multiplication, and division were illustrated using the M & M's. The idea of subtraction was illustrated by removing (eating) some of the candies and counting the remaining,

Gloria Sanok, (1980) has developed activities using saltine crackers to teach the concepts of congruent and similar. She also used the crackers to illustrate fractions. One big saltine cracker is equal to four smaller ones. One big cracker is also a bid square and can be divided into four smaller similar squares. She introduced the term quarter as being one-fourth of a whole cracker, which she later related to money.

One of the most widely used manupulative materials are the Cuisenaire Rods. Rosemary March describes Cuisenaire Rods as a set of sticks, each of a different color, which represents the numbers 1 through 10. The number one rod is a white cube. Each rod thereafter is one cube longer than the preceding one. The rods can be used for basic computation (add, subtract, multiply, and divide) and even algebra. March has found that teachers are very enthusiastic about Cuisenaire Rods because their students have improved significantly in their mathematical skills, especially the "special" children. Mathematics has become fun and on a level equal with their mental abilities.

The textbook, itself, does not supply ample motivation for the students to learn or for the teacher to teach. It is my responsibility, as a mathematics teacher, to create in my students, the desire to want to learn mathematics. It is also important that I be motivated to teach mathematics. To accomplish this task, I must believe in what I teach. Resourcefulness is also an important quality for a teacher. The use of supplemental tools along with the text can be an exciting adventure for the teacher and the students. The materials mentioned in the article is just a small sample of the fun resources available.

Mathematics is fun if the teacher uses more than just the paper and pencil routine. The manipulative mathematics materials allows the slowest student to grasp concepts that would otherwise be outside their grasp. Children arrive at school with the desire to learn. I am responsible as the teacher, to stimulate that desire. Only the teacher can put fun in her teaching.

REFERENCES

Liedtke, Werner. "Games for the Primary Grades". **Arithmetic Teacher.** Volume 28, Number 4, December, 1980. Pages 30 – 31.

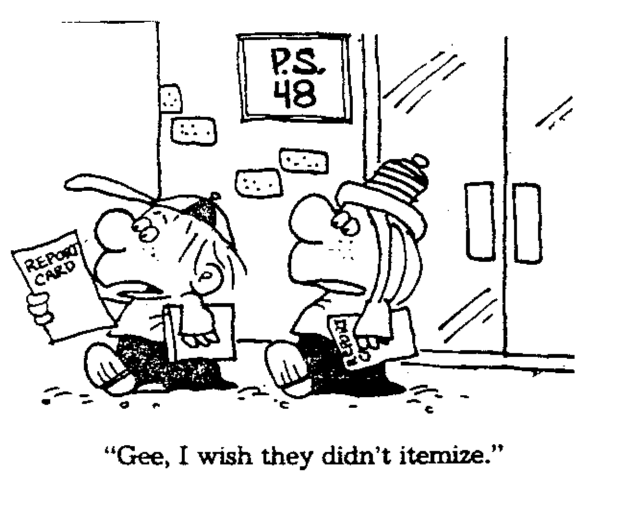
Dienes, Zoltan P. "Some Basic Processes Involved in Mathematics Learning". **Research in Mathematics Education.** 1967, pages 21 – 23, 26 – 28.

Wirtz, Robert W. "Looking at Children", **Banking on Problem Solving.** Pages 1 – 5.

Lunde, Henry M. "Sweeten Mathematics with M & M's". **Arithmetic Teacher.** Volume 28, Number 4. December, 1980, pages 16, 17.

Sanok, Gloria. "Mathematics and Saltine Crackers". **Arithmetic Teacher.** Volume 28, Number 4. December, 1980, page 36.

March, Rosemary. "George Cuisenaire and His Rainbow Rods", **Learning.** VII, November, 1977, pages 81 – 84.



"Gee, I wish they didn't itemize."

PYTHAGOREAN SYSTEMS OF NUMBERS

Sharon Arnold & Joseph Wiener
Pan American University

1. Introduction

The task of identifying all solutions in the positive integers of the equation

$$x^2 + y^2 = z^2 \qquad (1)$$

was raised in the days of the Babylonians and was a favorite with the ancient Greek geometers. Pythagoras himself has been credited with a formula for infinitely many such triples namely

$$x = 2n + 1, \quad y = 2n^2 + 2n, \quad z = 2n^2 + 2n + 1$$

where n is an arbitrary positive integer. This formula does not account for all right triangles with integral sides and it was not until Euclid wrote his Elements that a complete solution to the problem appeared. The characterization of all primitive Pythagorean triples is fairly straightforward: all solutions of Eq. (1) satisfying the conditions

$$\gcd(x, y, z) = 1, \quad 2 \mid x, \quad x > 0, \quad y > 0, \quad z > 0$$

are given by the formulas

$$X = 2st, \quad y = s^2 - t^2, \quad z = s^2 + t^2 \qquad (2)$$

for integers $s > t > 0$ such that gcd $(s,t) = 1$ and $s \not\equiv t \pmod 2$. Nonetheless, the interest in the study of new ways of generating Pythagorean triples (Schaumberger, 1982), (Schwartz, 1982), (Didomenico, 1983), (Ryden, 1983), or quadruples (Schaumberger, 1981) continues, and numerous papers appear in the field.

The pupose of this article is two-fold. First, without resorting to formulas (2) we derive intrinsic properties of Pythagorean numbers which are, of course, well known. However, our method of proof and the level of explanation are accessible to high-school students for, we do not use the theory of congruences. In the second part of the paper we develop a new elementary method of constructing Pythagorean systems with any number of terms. This approach is rather attractive since we can compose, in particular, such equalities

$$X_1^2 + x_2^2 + x_3^2 + \dots + x_n^2 = y_n^2$$

in which the sum of the first two squares is a square, the sum of the first three squares is a square, etc.

2. Properties of Pythagorean Numbers

In the simplest relation

$$3^2 + 4^2 = 5^2,$$

one of the numbers on the left is even and the other is odd; one of these numbers is a multiple of 3; one of them is divisible by 4; one of the three numbers is a multiple of 5. The student can prove that these remarkable properties are inherent in each primitive Pythagorean triple.

Property 1. In each primitive Pythagorean triple (1) either x or y is even, and the other is odd.

Indeed, both x and y cannot be even for, then z would also be even, and the triple is not primitive. Assuming that both x and y are odd numbers, we have

$$x = 2k + 1, \qquad y = 2m + 1$$

$$x^2 = 4k^2 + 4k + 1, \qquad y^2 = 4m^2 + 4m + 1,$$

where k and m are natural, and

$$z^2 = 4(k^2 + m^2) + 4(k + m) + 2.$$

From here we conclude that z is an even number and z^2 gives remainder 2 when divided by 4. This is impossible because the square of any even number is a multiple of 4.

Property 2. In each Pythagorean triple (1) either x or y is a multiple of 3.

Assuming the opposite, we have

$$x = 3k + 1 \quad \text{or} \quad x = 3k + 2$$

$$y = 3m + 1 \quad \text{or} \quad y = 3m + 2$$

Squaring these numbers

$$x^2 = 9k^2 + 6k + 1 \quad \text{or} \quad X^2 = (9k^2 + 12k + 3) + 1$$

$$y^2 = 9m^2 + 6m + 1 \quad \text{or} \quad y^2 = (9m^2 + 12m + 3) + 1 \qquad (3)$$

shows that the square of any number which is not a multiple of 3 gives the remainder 1 when divided by 3. On the other hand, from (1) and (3) it follows that

$$x^2 = 3q + 2$$

This means that z is not a multiple of 3, and that the remainder of z^2 divided by 3 is always 2 which is impossible.

Property 3. In each primitive Pythagorean triple (1) either x or y is a multiple of 4.

According to the first property, we may assume that in (1) x is even and y is odd. Then

$$x = 2k, \quad y = 2m + 1, \quad z = 2n + 1$$

and

$$(2k)^2 = (2n + 1)^2 - (2m + 1)^2.$$

By factoring the right-hand side we get

$$k^2 = (n - m)(n + m + 1).$$

If m and n are both even or both odd, then n - m is even; hence, k is an even number. If one of the numbers n or m is even and the other is odd, then n + m + 1 is even; consequently, k is also an even number. Thus, x = 2k is a multiple of 4.

Property 4. In each Pythagorean triple (1) either x, y or z is a multiple of 5.

Evidently, a number which is not a multiple of 5, when divided by 5, gives one of the following remainders: 1, 2, 3, 4. Then we verify that its square can give in remainder only 1 or 4. If x or y is a multiple of 5, the property is established. If we assume that in Eq. (1) both x and y are not multiples of 5, and both x^2 and y^2 have remainder 1, then z^2 gives remainder 2 when divided by 5 which is impossible. If one of the numbers x^2 or y^2 generates remainder 1 and the other gives remainder 4, then z is a multiple of 5. Finally, the case when both x^2 and y^2 yield the remainder 4 is impossible because z^2 has remainder 3.

3. Quadruples and Other Systems

At first, the problem of constructing quadruples of Pythagorean numbers might seem harder and more complicated than the original task of finding Pythagorean triples. In fact, it is by far easier to solve the equation with four unknowns

$$x^2 + y^2 + z^2 = w^2 \qquad (4)$$

than to satisfy Eq. (1) with three variables. The reason: Eq. (4) provides for the solver higher degree of freedom in the selection of variables. Indeed, we can choose two variables x and y almost arbitrarily, and then find z and w from (4). At the same time, this flexibility of choice means that Pythagorean quadruples possess fewer interesting properties than triples.

Property 1. In each primitive quadruple (4) two of the numbers x, y and z are even. If we assume that all three numbers x, y, and z are odd then w is also odd, that is,

$$x = 2k + 1, \quad y = 2m + 1, \quad z = 2n + 1, \quad w = 2q + 1.$$

Substituting these numbers in (4) gives

$$(2k+1)^2 + (2m+1)^2 + (2n+1)^2 = (2q+1)^2,$$

$$4(k^2 + m^2 + n^2 + k + m + n) + 2 = 4(q^2 \quad + q),$$

which is impossible because the right-hand side is a multiple of 4, whereas the left side is not. Suppose that x and y are odd and z is even, then w is even, and we have

$$x = 2k + 1, \quad y = 2m + 1, \quad z = 2n, \quad w = 2q.$$

In this case Eq. (4) takes the form

$$(2k+1)^2 + (2m+1)^2 + 4n^2 = 4q^2,$$

or

$$4(k^2 + m^2 + n^2 + k + m) + 2 = 4q^2,$$

which is again impossible for the same reason as above.

Property 2. In each quadruple either x, y, z or w is a multiple of 3.

Indeed, if we assume that none of x, y, and z is a multiple of 3, then

$$x^2 = 3k + 1, \quad y^2 = 3m + 1, \quad z^2 = 3n + 1$$

and

$$w^2 = 3(k + m + n + 1)$$

which means that w is a multiple of 3.

Now, we proceed to the construction of Pythagorean quadruples. According to Property 1, take for x and y two arbitrary numbers, one odd and the other even, and find an even number z and odd number w satisfying (4). Represent x^2 and y^2 as a product of two different factors:

$$x^2 + y^2 = st, \quad (s, \ t \geqslant 1, \ \ s > t).$$

If $x^2 + y^2$ is a prime, one of the factors equals 1. Then from (4) it follows that

$$w^2 - z^2 = st,$$

and we consider the system of equations

$$w + z = s, \quad w - z = t$$

which gives

$$z = (s - t) \ / \ 2, \quad w = (s + t) \ / \ 2. \tag{5}$$

Example 1. Take x = 1, y = 2; then $x^2 + y^2 = 5$. We write

$$x^2 + y^2 = 5 \cdot 1,$$

that is, s = 5, t = 1. Hence, from (5) we obtain z = 2, w = 3. This procedure yields the Pythagorean quadruple

$$1^2 + 2^2 + 2^2 = 3^2.$$

Example 2. Take x = 3, y = 4; then $x^2 + y^2 = 25$. We write

$$x^2 + y^2 = 25 \cdot 1$$

and put s = 25, t = 1. From (5) we have z = 12, w = 13. Thus,

$$3^2 + 4^2 + 12^2 = 13^2.$$

This example is of special interest in the sense that the sum of the first two squares is a square and the sum of the three squares is a square. We may continue this procedure to find Pythagorean systems containing more than 4 numbers.

Example 3. To find a system

$$x^2 + y^2 + z^2 + w^2 = k^2$$

in which the sum of the first two squares and the sum of the first three squares is a square we take two Pythagorean numbers, say, x = 3, y = 4 and have $3^2 + 4^2 = 5^2$. After this, we solve the equation

$$5^2 + z^2 = q^2.$$

Write

$$q + z = 25, \quad q - z = 1$$

whence

$$z = 12, \quad q = 13,$$

and

$$3^2 + 4^2 + 12^2 = 13^2.$$

Then we solve the equation

$$13^2 + w^2 = k^2$$

putting

$$k + w = 169, \quad k - w = 1.$$

From here,

$$k = 85, \quad w = 84.$$

Finally, we obtain

$$3^2 + 4^2 + 12^2 + 84^2 = 85^2.$$

Example 4. Sometimes the discussed method yields more than one Pythagorean quadruple. Take x = 1, y = 8; then $x^2 + y^2 = 65$. Solve the equation

$$w^2 - z^2 = 65.$$

The number 65 may be factored as 13 ˙ 5 or 65 ˙ 1. In this case we have two systems of equations:

$$w + z = 13 \qquad w + z = 65$$

and

$$w - z = 5 \qquad w - z = 1$$

with the solutions

$$z_1 = 4 \qquad z_2 = 32$$

$$w_1 = 9 \qquad w_1 = 33$$

Therefore we obtain two Pythagorean quadruples:

$$1^2 + 8^2 + 4^2 = 9^2 \text{ and } 1^2 + 8^2 + 32^2 = 33^2.$$

REFERENCES

Didomenico, Angelo S. "Eureka! Pythagorean Triples from the Multiplication Table." Mathematics Teacher 76 (January 1983): 48-51.

Ryden, Robert. "Nearly Isosceles Pythagorean Triples." Mathematics Teacher 76 (January 1983): 52-56.

Schaumberger, Norman. "A Classroom Approach to $x^2 + y^2 + z^2 = w^2$." Two-Year College Mathematics Journal 12 (November 1981): 331-332.

______. A Classroom Approach to Pythagorean Triples." Two-Year College Mathematics Journal 13 (January 1982): 61-63.

Schwartz, Helga. "Pythagorean Properties." The AMATYC Review, 4 (Fall/Winter 1982)" 26-29.

MATHEMATICS EDUCATION FOR ALL ARE WE SELLING OUR PRODUCT?

Albert K. Wiswell
Oaks Treatment Center
Brown Schools
Austin, Texas

Charles E. Lamb
The University of Texas at Austin
Austin, Texas

We need to look at how we, as mathematics educators, sell our product — namely, mathematics education. Are we selling our product effectively? Given the declining number of qualified teachers in the field, apparently not. The teaching of mathematics is the selling of a product. Our product is knowledge and skills.

In mathematics classes, we ask children to "spend" time, "Pay attention, and "expend" effort in order to "buy" our product. The sale is often a tough one. Many times, the benefits and rewards gained by a purchase are not realized for several years. Let's look at approaches used by other sales people as they marked their products.

First, we must establish a clear-cut need for the product. In today's society, there is an increasing demand for a highly-trained, technically-oriented populace which can meet the needs of industry. This fact is reflected in the types of job which are becoming available and the higher salaries associated with them. Even for those people not aiming toward this kind of employment, there is a need for mathematics literacy which will allow the management of one's time and money in an efficient fashion. Useful skills seem even more important in light of our present computer age and changing economy.

Second, we must induce potential customers to make the purchase. This task often takes the form of giving the prospective buyer a lot of individual attention. The buyer must be made to feel special; feel good about the purchase and about himself/herself. The buyer is then encouraged to express a need and desire for the product. In the mathematics classroom, we can strive for this "special" feeling by enabling students to be successful.

Success bolsters students' conficende and self-esteem. They will be able to see that effort produces results. Similarly, a challenge to succeed in a difficult endeavor can increase the commitment of effort. For example, we might "impress" our clients by letting them study interesting topics of their choosing, out of the normal teaching sequence. As well as the enjoyment of new material, there develops a commitment to expend the effort to master it. Also, the student might realize gaps in his/her knowledge base and thus be motivated to acquire needed skills.

Third, the sale must be made. The salesperson has to be motivated to keep working on the sale and the buyer has to be motivated to buy. Teachers need constant support through inservice and other programs to keep their motivation high. Highly motivated teachers will be more likely to provide motivational activities for students. They will be more likely to supplement their class instruction with enrichment, games, technology, and humor.

More sales will mean more mathematically-capable students, a better enlightened citizenry and a larger pool of potential future teachers to sell mathematics teaching tomorrow. How many sales have you made today?

LOTTERY PROBABILITIES

Bonnie H. Litwiller and David R. Duncan
Professors of Mathematics
University of Northern Iowa
Cedar Falls, Iowa

Suppose that a lottery is conducted by a service club. If Tom wishes to play, he must do the following:

1. At the weekly meeting of the club, he purchases a card. Each card contains 24 distinct numbers randomly chosen from the set S where S = (1, 2, 3, ... 75).
2. Near the end of each meeting, the president of the club reads a set of 50 distinct numbers drawn randomly from S.
3. Tom compares the 24 numbers on his card with the 50 numbers read by the president.
4. If all 24 of Tom's numbers are contained in the 50 numbers read to the group, he is the winner.

The club agreed to denote half of the proceeds for each game to their charitable project? the other half is divided among the winners. If there are no winners, the entire proceeds are denoted to the project.

At each meeting new cards are distributed for a new game. Cards cannot be re-used.

What is the probability that Tom wins? To calculate this, note that the number of possibel 50-number combinations that can be read to the group is $\binom{75}{50} = \frac{75!}{50!25!}$. In order for Tom to win, the 50 numbers that are read must be taken from the following two sets:

1. All 24 of the numbers on Tom's card must be read. The number of ways of doing this is $\binom{24}{24} = 1$.
2. Exactly 26 of the 51 numbers not on Tom's card must be read. By the Fundamental Principle of Counting, the number of 50-number combinations that would cause Tom to win is $\binom{24}{24} \cdot \binom{51}{26} = 1$

$$\binom{24}{24} \cdot \binom{51}{26} = 1 \cdot \frac{51!}{26!25!} = \frac{51!}{26!25!}.$$

The probability of Tom's winning is thus:

$$\frac{\frac{51!}{26!25!}}{\frac{75!}{50!25!}} = \frac{51 \cdot 50 \cdot 49 \cdot \ldots \cdot 27}{75 \cdot 74 \cdot 73 \cdot \ldots \cdot 51} = .0000047.$$

Some of our students thought that Tom should have a 50—50 chance of winning since 24 is almost 1/2 of 50. What would your students predict? This actual probability is really quite small.

What is the probability that there is at least one winner after the 50 numbers are read? To answer this question, the number of players must be known. If Tom is the only player, the probability of there being a winner is .0000047. Tom, no doubt, is not the only player. If there were 100 players (a typical attendance at the club's weekly meeting), each of them has a probability of 1 — .0000047 = .9999953. The probability that there is at least one winner is 1 - (probability that no one wins). Since the probability that no one wins is $(.9999953)^{100}$, the probability that there is at least one winner is therefore $1 - (.9999953)^{100} = .000470$. In other words, if this game were played for 2130 weeks $\left(\frac{1}{.000470} = 2130\right)$, only 1 winner would be "expected."

Tom calls this information to the officers' attention. To improve the chances of there being a winner, it was decided to reduce the size of the set S. Recall that S =(1,2,3, . . .,75). By how many numbers should S be reduced to make the game more interesting?

The club treasurer, Gerald, is asked to compute the probability of having a winner for different sizes of S. The club is leaving unchanged the number of numbers on each game card (24) and the number of numbers read by the president (50). To do this, Gerald uses the following formula for S = (1,2,3, ... n).

$$P \text{ (Tom wins)} = \frac{\binom{24}{24} \cdot \binom{n-24}{26}}{\binom{n}{50}}$$

$$= \frac{\frac{(n-24)!}{26!\,[(n-24)-26\,]!}}{\frac{n!}{50!\,(n-50)!}}$$

$$= \frac{\frac{(n-24)}{26!(n-50)!}}{\frac{n!}{50!\,(n-50)\,!}}$$

$$= \frac{(n-24)\,!}{26!} \cdot \frac{50!}{n!}$$

$$= \frac{50 \cdot 49 \cdot 48 \cdot \cdots \cdot 27}{(n-23) \cdot (n-22) \cdot \cdots \cdot n}$$

Table I represents the results of the calculations. Because the club wished to make the probability of winning more likely, only values less than or equal to 75 are used for n. In each case the probability that there is at least one winner is calculated as $1 - [1 - P(\text{Tom wins})]^{100}$.

TABLE I

n	S	P (Tom wins)	P (at least one winner)
75	(1,2,3, • • •, 75)	.0000047	.00047
74	(1,2,3, • • •, 74)	.0000069	.00069
73	(1,2,3, • • •, 73)	.000010	.0010
72	(1,2,3, • • •, 72)	.000015	.0015
71	(1,2,3, • • •, 71)	.000023	.0023
70	(1,2,3, • • •, 70)	.000035	.0035
69	(1,2,3, • • •, 69)	.000053	.0053
68	(1,2,3, • • •, 68)	.000081	.0081
67	(1,2,3, • • •, 67)	.00012	.012
66	(1,2,3, • • •, 66)	.00019	.019
65	(1,2,3, • • •, 65)	.00031	.031
64	(1,2,3, • • •, 64)	.00048	.047
63	(1,2,3, • • •, 63)	.00078	.075
62	(1,2,3, • • •, 62)	.0012	.11
61	(1,2,3, • • •, 61)	.0020	.18
60	(1,2,3, • • •, 60)	.0034	.29
59	(1,2,3, • • •, 59)	.0056	.43
58	(1,2,3, • • •, 58)	.0095	.62
57	(1,2,3, • • •, 57)	.016	.80
56	(1,2,3, • • •, 56)	.028	.94
55	(1,2,3, • • •, 55)	.049	.993
54	(1,2,3, • • •, 54)	.087	.9999
53	(1,2,3, • • •, 53)	.16	1.0000
52	(1,2,3, • • •, 52)	.28	1.0000
51	(1,2,3, • • •, 51)	.53	1.0000
50	(1,2,3, • • •, 50)	1.000	1.0000

The club agrees that there should be a winner about half the time so they agree to select numbers from the S = (1,2,3, • • • 59).

Tom's friend John notes that 50 of the 59 numbers are read? that is, 85% of the numbers from S are read. John asks, "Does this mean that there should be a winner 85% of the time?" How should Tom answer this?

John also notes that at the beginning of Table I there is a short cut method to go from column 3 to column 4 — just multiply by 100. Why does John's method appear to work at the beginning of the table but fail near the end of the table?

From Table I, it can be seen that Tom will win just over 1/2 of 1% of the time (about once every four years). It seemed to some of our students that Tom should win more frequently. What would your students think?

Challenges to the reader and his/her students:

The club decided to change the rules of the game in other ways.

What effect will each of them have on the probability that Tom wins and the probability that there is at least one winner?

1. Suppose the game is played with a larger group of people.
2. Suppose that players could purchase cards with different numbers of numbers on them. Suppose, for example, that instead of 24 numbers, the cards contained 22 numbers.
3. Suppose that if no winner is found after the first 50 numbers are read, additional numbers are read until there is a winner. Find the probabilities resulting from **each** additional number that is read.

SOLVING RADICAL EQUATIONS

by John Huber and Joseph Wiener
Pan American University

Most students have difficulty solving radical equations. The method used in most high school algebra texts (Dolciani, Sorgenfrey, Wooten, and Kane, 1977; Foerster, 1980; and Travers, Dalton, and Brunner, 1981) involves isolating a radical and squaring, cubing, etc., both sides of the equation. If a radical remains, repeat the process. Then solve the equation, check the solutions, and disregard the extraneous roots. The purpose of this paper is to describe a method for solving radical equations which transforms the radical equation to a system of equations.

EXAMPLE 1:

$$\sqrt{x+4} = x - 8.$$

Let

$$y = \sqrt{x+4} \qquad (1)$$

Then $y^2 = x + 4$ where $y \quad 0$. The original equation

$\sqrt{x+4} = x - 8$ becomes

$$y = x - 8 \qquad (2)$$

$$y^2 = x + 4. \qquad (3)$$

Subtracting (2) from (3) we have

$$y^2 - y = 12$$

or

$$y^2 - y - 12 = 0$$

from which

$$(y - 4)\ (y + 3) = 0. \qquad (4)$$

Since $y \quad 0$ the only solution to (4) is $y = 4$. Then solving (2) with $y = 4$, yields $x = 12$.

Observe that in solving for only the non-negative roots of (4) no extraneous roots are introduced.

EXAMPLE 2:

$$\sqrt{2x-3} - \sqrt{x-2} = 1$$

Let

$$y = \sqrt{2x-3}$$

and

$$z = \sqrt{x-2}.$$

Then

$$y - z = 1, \qquad (5)$$

$$y^2 = 2x - 3, \text{ and} \qquad (6)$$

$$z^2 = x - 2. \qquad (7)$$

Subtracting twice (7) from (6) gives

$$y^2 - 2z^2 = 1. \qquad (8)$$

Solving (5) for y and substituting into (8) yields

$$y^2 - 2(y-1)^2 = 1$$

or

$$y^2 - 4y + 3 = 0 \qquad (9)$$

Factoring (9) we have

$$(y - 3)\ (y - 1) = 0$$

from which

$$y = 1 \text{ or } y = 3.$$

Then

$$1 = \sqrt{2x-3} \quad \text{implies} \quad x = 2$$

and

$$3 = \sqrt{2x-3} \quad \text{implies} \quad x = 6.$$

Thus the solution set of $\sqrt{2x-3} - \sqrt{x-2} = 1$, is (2, 6) .

EXAMPLE 3:

$$x - \sqrt{3x-2} = 4.$$

Let

$$y = \sqrt{3x-2}. \quad \text{Then}$$

$$x - y = 4 \qquad (10)$$

$$y^2 = 3x - 2. \qquad (11)$$

Solving (10) for x and substituting into (11)

$$y^2 = 3(y + 4) - 2.$$

Simplifying

$$y^2 - 3y - 10 = 0.$$

Factoring we have

$$(y - 5)\ (y + 2) = 0$$

from which

$$y = 5 \text{ or } y = -2.$$

Since $y \quad 0$, the only solution is $y = 5$. Thus

$5 = \sqrt{3x-2}$ implies $x = 9$.

EXAMPLE 4:

$$\sqrt{x^2 + 5x - 6} + \sqrt{x^2 + 3x - 3} = 1.$$

Let

$$y = \sqrt{x^2 + 5x - 6},$$

$$z = \sqrt{x^2 + 3x - 3},$$

Then

$$y + z = 1,$$

(continued on page 11)

Deriving the Quadratic Formula Without Completing the Square

Bella Wiener
Pan American University

Recently Dobbs (1982), Wiener (1982), Huber and Wiener (1983) presented a method for deriving the quadratic formula without "completing the square." The purpose of this paper is to obtain the quadratic formula by means of an appropriate substitution. This approach has several advantages in comparison with the traditional method: we avoid the procedure of completing the square, the translation of coordinates which we employ is very helpful in sketching graphs of quadratic trinomials, the idea is useful also in the theory of higher - order equations.

To solve the quadratic equation

$$ax^2 + bx + c = 0, \qquad (1)$$

we make the substitution

$$x = h + u \qquad (2)$$

where u is a new variable and h is a constant which will be indicated later.
Sustituting (2) in (1) gives

$$a(h+u)^2 + b(h+u) + c = 0,$$

$$au^2 + (2ah + b)u + (ah^2 + bh + c) = 0 \qquad (3)$$

Now we require that the coefficient of u in (3) equals zero:

$$2ah + b = 0.$$

Hence,

$$h = -\frac{b}{2a}, \qquad (4)$$

and we arrive at the conclusion that the substitution

$$x = -\frac{b}{2a} + u \qquad (5)$$

reduces (1) to the simplest form (3) in which the coefficient of u is zero. To finish the derivation of the quadratic formula, we substitute (4) in (3) and get

$$au^2 + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right) = 0,$$

$$u^2 = \frac{b^2 - 4ac}{4a^2}, \quad u = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

It remains to substitute these values of u in (5) which yields the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Dobbs, David E. "Discovering the Quadratic Formula". ILLINOIS MATHEMATICS TEACHER May, 1982. 27 - 28.

Wiener, Bella. "Another Quadratic Formula". MATHEMATICS TEACHING IN COLLEGE Fall, 1982. 21 - 24.

Huber, John and Wiener, Bella. "Another Derivation of the Quadratic Formula". ILLINOIS MATHEMATICS TEACHER (to appear).

SOLVING RADICAL EQUATIONS

(Continued from page 10)

$$y^2 = x^2 + 5x - 6, \qquad (12)$$

$$z^2 = x^2 + 3x - 3,$$

$$y^2 - z^2 = 2x - 3,$$

$$(y - z)\ (y + z) = 2x - 3,$$

$$y - z = 2x - 3,$$

$$y + z = 1$$

$$2y = 2x - 2$$

$$y = x - 1 \qquad (13)$$

Substituting (13) into (12) we have

$$(x-1)^2 = x^2 + 5x - 6$$

$$x^2 - 2x + 1 = x^2 + 5x - 6$$

$$x = 1$$

This technique can also be used very effectively in the derivation of the equations of the ellipse and hyperbola. For the derivation of the equation of the ellipse using this technique see Huber & Wiener (to appear).

REFERENCES

Dolciani, M.P.; Sorgenfrey, R.H.; Wooten, W.; and Kane, R.B. ALGEBRA AND TRIGONOMETRY: STRUCTURE AND METHOD BOOK 2. Houghton Mifflin Company, 1977.

Foerster, P. ALGEBRA AND TRIGONOMETRY: FUNCTIONS AND APPLICATIONS. Addison-Wesley Publishing Company, 1980.

Huber, J.C. and Wiener, J. "Deriving the Equations of the Ellipse and Hyperbola." TWO-YEAR COLLEGE MATHEMATICS JOURNAL, (to appear).

Travers, K.J.; Dalton, L.C.; and Brunner, V.F. USING ADVANCED ALGEBRA (Third Edition). Laidlaw Brothers Publishers, 1981.

PLEASE SOLICIT NEW MEMBERSHIPS!

PROFESSIONAL MEMBERSHIP APPLICATION

Date: ______ School: ______ School Address: ______

Position: ☐ teacher, ☐ department head, ☐ supervisor, ☐ student,* ☐ other (specify) ______

Level: ☐ elementary, ☐ junior high school, ☐ high school, ☐ junior college, ☐ college, ☐ other (specify ______

Other information ______

		Amount Paid
Texas Council of Teachers of Mathematics	☐ New membership ☐ Renewal membership	5.00
Local ORGANIZATION: ______	☐ New membership ☐ Renewal membership	
OTHER: ______	☐ New membership ☐ Renewal membership	

Name (Please print) ______ Telephone ______

Street Address ______

City ______ State ______ ZIP Code ______

National Council of Teachers of Mathematics		
	Check one: ☐ New membership ☐ Renewal membership	
	$30.00 dues and one journal ☐ Arithmetic Teacher or ☐ Mathematics Teacher	
	$40.00 dues and both journals	
	$15.00 student dues and one journal* ☐ Arithmetic Teacher or ☐ Mathematics Teacher	
	$20.00 student dues and both journals*	*Note New Membership and Subscription Fees*
	12.00 additional for subscription to *Journal for Research in Mathematics Education* (NCTM members only)	
	The membership dues payment includes $10.00 for a subscription to either the *Mathematics Teacher* or the *Arithmetic Teacher* and 75¢ for a subscription to the *Newsletter.* Life membership and institutional subscription information available on request from the Reston office.	

*I certify that I have never taught professionally ______ *(Student Signature)* *Enclose One Check for Total Amount Due* →

Fill out, and mail to Mrs. Bettye Hall, 3830 Richmond, Houston, Texas 77027

NOW!

TEXAS MATHEMATICS TEACHER
J. William Brown, Editor
Texas Council of
Teachers Of Mathematics
Woodrow Wilson High School
100 S. Glasgow Drive
DALLAS, TEXAS 75214

NON-PROFIT ORGANIZATION
U. S. Postage
Paid
Dallas, Texas
Permit #4899