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TEXAS MATHEMATICS TEACHER

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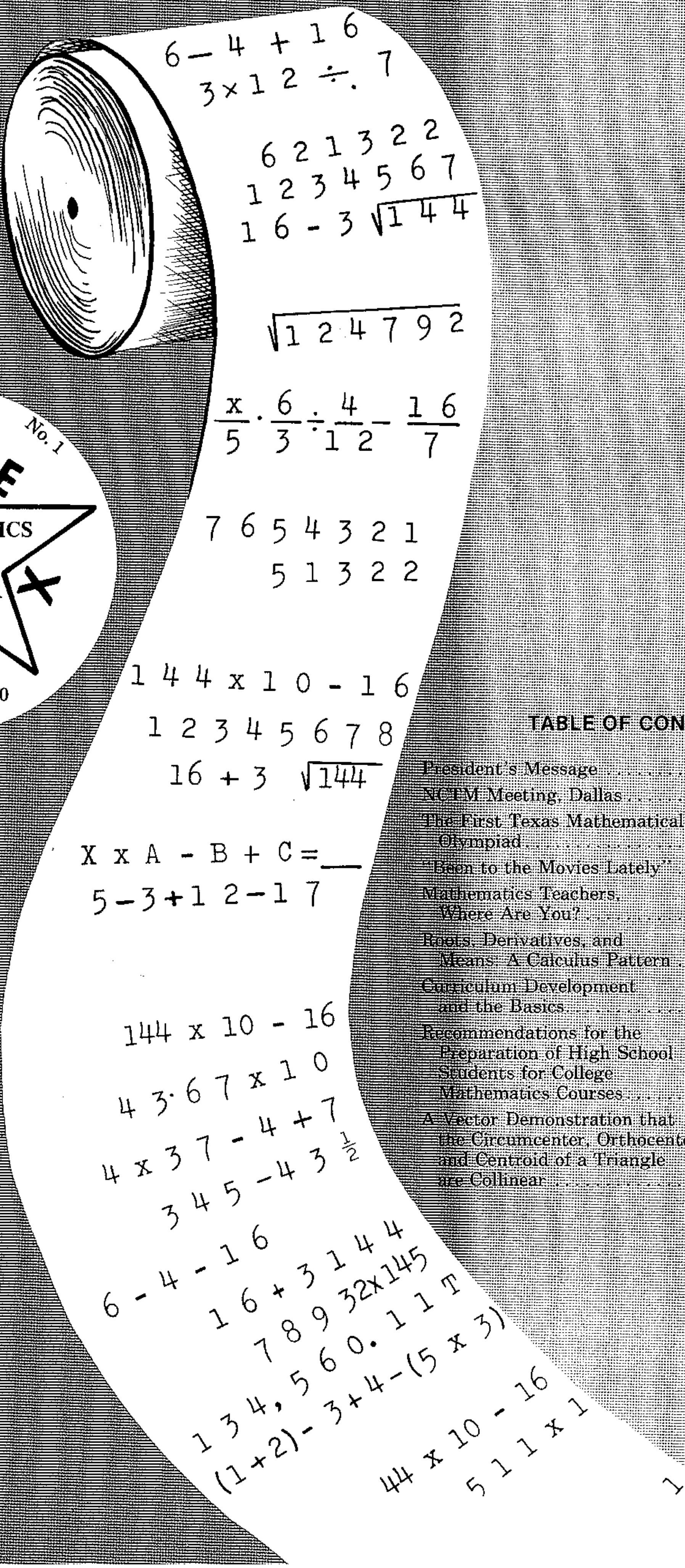

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TEXAS MATHEMATICS TEACHER is the official journal of the Texas Council of Teachers of Mathematics. The views expressed are the contributor's own and are not necessarily those of the publisher or the editor. All manuscripts and correspondence about this publication should be addressed to Mr. J. William Brown, *Texas Mathematics Teacher,* 100 So. Glasgow Drive, Dallas, Texas 75214.

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PRESIDENT'S MESSAGE

Happy New Year!

I hope that each of you had a happy holiday season and came back to work refreshed for the days ahead.

The recent election of officers for TCTM has added the following personnel to our Executive Board. They are President-elect, Mrs. Patsy Johnston who teaches at Southside High in Fort Worth; Vice-president, Dr. Floyd Vest, North Texas State University; and Treasurer, Gordon Nichols who teaches in the middle school in Pasadena. Congratulations to each of these newly elected officers.

Since there will not be another issue of the Journal before the Name-of-Site meeting in Dallas, March 6 - 8, I am taking this opportunity to urge each of you to make plans to attend this convention. Program booklets have been mailed to NCTM members; however, if you did not receive a copy contact the NCTM office and one will be sent to you. The address is 1906 Association Drive, Reston, Virginia 22091.

I look forward to seeing many of you at this meeting. We plan to have a short TCTM business meeting at sometime during the convention. Look for the announcement of the time and place in the program supplement and in the registration area.

Did you receive your copy of the Newsletter in December? We plan to publish another issue in April; so please send me any ideas, activities, or teaching strategies that have been successful in your classroom.

An increase in membership is always a goal of any organization and TCTM is no exception. Encourage other mathematics teachers in your building to become members of TCTM. Thank you for your help and cooperation.

Anita Priest

NCTM MEETING — DALLAS
MARCH 6 - 8, 1980

Dallas, a city rich in history, progressive in education and cultural arts, exciting with sports and recreation, and exhilarating with bustling trade and commerce, will host the 6 - 8 March NCTM meeting. The Greater Dallas Council of Teachers of Mathematics welcomes you to enjoy the hospitality of Big D.

Featured speakers will include NCTM President Shirley Hill, John McKetta, Shirley Frye, Richard Brown, Richard Andree, and Harold Jacobs. Extend your mathematics with James Margenau in the mathematics lecture series. Section meetings and workshops have been designed to interest teachers at the kindergarten through two-year college levels.

Plan now to savor the flavors of Texas, where you can "ya hoo" at a rodeo, muse at the symphony, or "smooth move" at the disco.

Hotel rooms have been reserved for registrants at the Sheraton-Dallas Hotel, where all convention activities will be held. Your room requests must be received no later than 15 February 1980. The program will begin on Thursday, 6 March, at 12:45 p.m. and end on Saturday, 8 March, at 1:00 p.m.

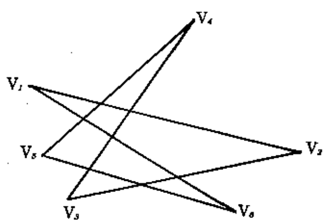
The First TEXAS MATHEMATICAL OLYMPIAD

George Berzsenyi

Lamar University

Sponsored by the Texas Section of the Mathematical Association of America, the first Texas Mathematical Olympiad (TMO) took place on March 24, 1979. It was a two-hour contest for those students who scored 80 or more points on the Annual High School Mathematics Examination held on March 6, 1979. A total of 229 students took part in the TMO, representing 68 high schools of Texas. The problems posed to the students were as follows:

PROBLEM 1: As shown in the figure below, a crossed hexagon with vertices $V_1, V_2, \ldots, V_6$ can cross itself 7 times.



Furthermore, it can be shown that 7 is the maximal number of self-crossings. What is the corresponding maximal number of self-crossings for a crossed heptagon (polygon with 7 vertices)? Prove that your result is indeed maximal.

PROBLEM 2: Show that no number of the form $(2n-1)(6n-1)$, where n is a positive integer, can be the square of an integer.

PROBLEM 3: Let $f(x)=1979-\frac{1}{x-1978}$ and define $f^{(2)}(x)=f(f(x))$, $f^{(3)}(x)=f(f(f(x)))$, etc. Calculate $f^{(1980)}(1979)$.

PROBLEM 4: Three sisters went to the market to sell their chickens. The first sister had 10 chickens to sell, while the second one had 16 and the third one had 26. During the morning hours they each sold some, but not all, of their chickens for the same price per chicken. During the afternoon the sisters uniformly lowered the price per chicken and sold them all. At the end of the day each of the sisters collected \$35. What was the price of each chicken sold in the morning?

The students were asked not only to solve the problems but to provide extensions, generalizations, alternate solutions, etc., whenever possible. Problem 3 proved to be the easiest; most students succeeded in showing that $f^{(2)}(1979)$ is not defined and hence $f^{(1980)}(1979)$ cannot be evaluated. Several solvers pointed out that if $x\neq1978$ or 1979, then for each $n=1, 2, \ldots$ $f^{3}(x)=x$; some of these solvers showed that the same holds for the exceptional values of x, provided that the domain and range of f include a point at infinity. Problem 1 proved to be the next most popular, allowing several methods of attack. Most solvers found the desired answer (14) by a step by step construction, counting the maximal number of possible self-crossings at each step. Others observed that each side can cross at most four others and hence at most $(7\times4)/2$ self-crossings are possible. This lead them to generalize their solutions to crossed-polygons with an odd number of vertices. Problem 2 was attempted by many but solved by only a very few students. Many claimed that the resulting square would have to be of the form $(an+b)^2$ implying that $a^2=12$, which can't hold for integers a. Unfortunately, this approach works only in case of identities. Others let $k^2=(2n-1)(6n-1)$, solved the resulting quadratic equation in n, and considered its discriminant, $16(3k^2+1)$. Unfortunately, it is a perfect square for infinitely many k's, whose general form is not easy to find. Consequently they could not complete their proof and show that nevertheless, the resulting n cannot be an integer. The easiest approach to Problem 2 hinged on the fact that $2n-1$ and $6n-1$ are relatively prime, hence if their product is a perfect square, they each must be perfect squares, but $6n-1$ cannot be the square of an integer. Alternately, $2n-1=a^2$ and $6n-1=b^2$ yield $b^2=a^2+2$, which can be shown to be impossible by mod 3 considerations. Problem 4 proved to be the most difficult, only a handful of students succeeded to derive the morning price of \$3.75. Several found upper and lower bounds for the answer, which lead them to educated guessing, a highly credible aspect of problem solving. Unfortunately, most students got lost in the system of equations they kept complicating rather than simplifying.

As none of the students succeeded in solving all four problems, it was decided that the First Prize would not be awarded this year. The five students earning second and third prizes solved three of the problems, making up for minor shortcomings with elegance, insightful remarks and generalizations. In alphabetical order — the winners of Second Prizes were:

Paul Baer (Westbury HS, Houston)
Ferrell S. Wheeler (Forest Park HS, Beaumont);

the winners of Third Prizes were:

Pang-Chieh Chen (S. H. Rider HS, Wichita Falls)
David Chenevert (Memorial HS, Houston)
Mike Storey (Sharpstown HS, Houston).

In addition to the above, the following students are commended for having solved at least two problems, providing extensions whenever possible:

Ted Biggs (Westchester HS, Houston), Donald Callender (Westbury HS, Houston), Derek Clegg (R. L. Paschal HS, Fort Worth), Joe Dellinger (Bishop Dunne HS, Dallas), Reese Faucett (Spring Woods HS, Houston), Melissa Ann Geiger (Robert E. Lee HS, San Antonio), Katherine Hoppe (Clear Creek HS, League City), John Kennedy (Clear Lake HS, Houston), Mike Klein (Strake Jesuit College Prep, Houston), Jeannie Kocurek (S. F. Austin HS, Austin), Gregg Lesartre (Memorial HS, Houston), Mark Needels (Jesse Jones HS, Houston), Sharon Kay Tatge (Warren Travis White HS, Dallas), Jonathan Weitsman (A&M Consolidated HS, College Station).

It is of interest to note that of the nineteen students named above, 9 are seniors, 9 are juniors and 1 is a sophomore. In particular, both of the winners of second prizes and two of the third prize winners are juniors. Three of the top five students scored only in the 80's in the Annual High School Mathematics Examination, while the other two were among the top four scorers in Texas (with scores of 119 and 117, respectively). Among those winning Honorable Mention (commended above), seven scored in the 80's, a clear indication of the fact that creative problem solvers don't necessarily do well on objective-type tests.

I believe that the contest was most successful in identifying some of the best problem solvers, and I am very appreciative of the opportunity to assist in accomplishing this. Congratulations are due to the winners and thanks to their dedicated teachers for their invaluable cooperation.

I would also like to thank the members of the Texas Mathematical Olympiad Committee, Dr. Robert Greenwood, Mr. William McNabb and Dr. James R. Boone, for their dedicated work as well as members of Lamar's Department of Mathematics, Professors Richard A. Alo', Michael Laidacker, Charles Lauffer, Jack Mades, David Read, Jerry Stark and Sam Wood for their invaluable assistance in the evaluation of the solutions submitted. The trophies were presented to the winners by Professors Louis Huffman, Margaret Hutchinson, Robert Thrall, James Younglove and the author during the respective school honors ceremonies. Problem 4 was "borrowed" from a Russian source.

The second Texas Mathematical Olympiad will be held on March , 1980 during the first two class-periods. Otherwise the format will remain the same.

"Been to the Movies Lately?"

By Charles E. Lamb

The University of Texas at Austin

Although the film has been available for instructional use for many years, one wonders, "To what extent is it used?" On talking to teachers of mathematics, it appears that the answer is *very little.* This seems to be the case no matter the grade level (elementary, middle, or secondary). The purpose of this brief article is to remind teachers of some valuable ways to utilize films for instructional purposes in the mathematics classroom.

Clearly the novelty of having a film in class, could produce motivational gains for many kinds of students. Children with "hate" for mathematics may begin to see that is can indeed be fun. Using films could also be motivational for the teacher. Wouldn't it be enjoyable to see kids having a good time in a mathematics class?

In school emphasis should be place on the 3R's. In this case, let's look at 2R's of film use: Reinforce and Review. Before we use a film to reinforce a concept, we have to introduce the idea. Films could serve this function. Then, in the middle of the unit, another film could reinforce previously learned ideas. Films could also help us to close out a topic by reviewing ideas before a test.

Finally, keeping a supply of films on hand could serve as a back-up or fill-in for teachers. The last school day before a vacation or the days of special activity such as a party or an assembly might be a place to use this strategy.

Teachers in Texas are fortunate to have Regional Service Centers (ESC's) at their disposal. Take a few minutes and stop by yours. Get acquainted with their film service (catalogues and film libraries). Then, do as I do with my classes. Make some popcorn and go to the movies.

Mathematics Teachers, Where Are You?

Donald J. Dessart

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A headline in the *Washington Post* of May 28, 1979, announced "Math Teacher Recruiting A Problem". The article went on to describe the plight of the head of the Mathematics Department at Bethesda-Chevy Chase High School, a prestigious Washington, D. C. suburban school, in attempting to hire a much needed algebra teacher. There were eight names on the list of applicants in the files of the Personnel office, but many of these had already found jobs or had previously left the area. After much searching, an algebra teacher was found, but the search led the Director of Personnel to remark, "There's a terrible myth going around that there is an oversupply of teachers!" He continued to explain that actually few people are seeking teaching positions in mathematics, science, industrial arts, or geography.

Kenneth Eskey, a Scripps-Howard staff writer, reported in the Knoxville, Tennessee *News Sentinel* of June 6, 1979, that "the depressed job market for school teachers is about to bottom out. It could evolve into a teacher shortage by the mid 1980's". He continued with the observation that "spot shortages are cropping up around the country in teaching fields like mathematics, science, industrial arts, and the education of physically and mentally handicapped children".

Is There Really A Shortage?

After the newspaper rhetoric dies down, one naturally wonders if there is indeed a shortage of mathematics teachers or whether this situation is only the result of overly zealous newspaper reporters. Consideration of recent statistical surveys shed some light on this question. We will look at such a survey.

The 1978 edition of *The Condition of Education,* a highly reliable statistical publication of the National Center for Education Statistics (NCES) of the U. S. Department of Health, Education, and Welfare, reported the results of a survey of a sample of 507 school districts. This study was designed to determine a national estimate of the number of unfilled teacher positions during the fall of 1977. It was found that the field of teaching learning disabled students experienced a shortage of 1500 teachers. This was followed by bilingual education with a shortage of 1200 teachers and mathematics with a need for 1100 teachers. A lack of 400 teachers was cited for the natural and physical sciences and such fields as art, business, foreign languages, home economics, and social studies reported unfilled teacher positions of less than 50 each. Consequently, it appears that the shortage of mathematics teachers is real and not solely the illusion of newspaper rhetoric.

What About Future Needs?

One might easily conclude that these districts were reporting temporary shortages without much regard to future needs. But that was not the case! This same study estimated the number of districts in the United States that would experience shortages during the next five years. As one might have expected, the field of teaching the learning disabled student led with a report that 1200 districts expected a future demand. This specialty was followed by demands for teachers of the gifted and talented (1000 districts) and mathematics (900 districts). Fields such as business, foreign languages, and distribution education would experience demands in fewer than 50 districts during the next five years

One might be inclined to suggest that perhaps there are sufficient numbers of propsective teachers in our colleges and universities to alleviate future shortages. An analysis of enrollments does not seem to support such a conjecture. The *Digest of Education Statistics, 1977-78* of NCES noted that in 1975-76, there was a total of 925,746 bachelor's degrees conferred in the United States. Of this total number, 1,358 or a mere .15 percent of the total were degree holders in mathematics education. It was not surprising to learn that 24,181 students (nearly 18 times the number of students in mathematics education) received degrees in physical education. When one realizes that a large number of the mathematics education graduates may never enter classrooms because of their loss to business of industry, it is easy to see that the number of graduates who enter mathematics teaching is very small.

Other statistics confirm the fear that the "pipeline" of mathematics teachers is drying up. Professor Alan Tucker of the State University of New York at Stony Brook examined the responses of a survey conducted by the prestigious Conference Board of Mathematical Sciences (CBMS). Tucker reported in the *CBMS Newsletter* of October, 1976 that the number of students electing upper division pure mathematics courses had declined by nearly 70 percent during the period of 1970-75. Since potential mathematics teachers study upper division courses before graduation, this decline should also be reflected in a downturn in mathematics teacher education enrollments. This finding is consistent with the trend noted in *The Digest of Education Statistics* discussed above.

What Can We Do?

If these statistics convince us that we face a shortage of qualified secondary school mathematics teachers, then we as a profession are obligated to take actions which will correct this situation. Shortages of mathematics teachers can only mean that the youth of our nation will be deprived of the best mathematics education at a time when our society needs mathematically oriented thinking to help solve complex environmental and energy problems. The lack of the availability of mathematics education in our secondary schools seriously limits the number of students who pursue scientific studies on the college level and eventually work on these problems. In particular, it is a well known fact that too many young women do not enter scientific fields because of their inadequate preparation in high school mathematics.

But what can we do? Below is a suggested *five* point program for action by mathematics teachers.

1. *Encourage* good mathematics students who also work well with people to consider seriously mathematics teaching as a career. Point out that the satisfactions of teaching can far outweigh the dissatisfactions, and that improvements can only occur if we have a strong, tough, dedicated corps of professionals.

2. *Work* to improve conditions for mathematics teachers by:
   - Making facts concerning the bad effects of shortages of qualified mathematics teachers known to decision makers—school board members, supervisors, principals, and superintendents.
   - Supporting professional associations that are working to improve and strengthen our profession.

3. *Enlist* the aid of others—students, parents, service organizations, local academies of science, in-

dustries—those who benefit most from sound mathematics education of our youth, to work for better teaching conditions, greater financial rewards, and greater recognition of the values of a good mathematics education.

4. *Seize* every opportunity to improve teaching to demonstrate that ours is a vital, active profession. Become acquainted with the latest uses of hand-held calculators, the kinds of concepts to emphasize when teaching the use of the metric system, and more efficient ways to develop skills.

5. *Show* that mathematics is a viable, useful study by emphasizing problem solving and the applications of mathematics to the real world. The 1979 Yearbook of NCTM *Applications in School Mathematics,* edited by Sidney Sharron should be of considerable help.

In conclusion, the shortage of mathematics teachers should stimulate us to improve our profession so that more young people will be attracted to our ranks. This is our profession, and we must be the leaders in attempting to improve it.

Roots, Derivatives, and Means: A Calculus Pattern

David R. Duncan and Bonnie H. Litwiller

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Cedar Falls, Iowa

Let f(x) be a quadratic polynomial function with real coefficients. Let a and b be the roots of the equation $f(x) = 0$ (a and b are either both real or are conjugate complex numbers.) f(x) may be written in the form $f(x) = k(x-a)(x-b)$. Using the product rule, the derivate is:

$$f'(x) = k[(x-a)\cdot 1 + (x-b)\cdot 1]$$
$$f'(x) = k[x-a+x-b]$$
$$f'(x) = k[2x-a-b]$$
$$f'(x) = k[2x-(a+b)].$$

Setting the first derivative equal to 0, we have:

$$k[2x-(a+b)] = 0. \text{ Thus, } x = \frac{a+b}{2}.$$

The root of $f'(x) = 0$ is the arithmetic mean of the roots of $f(x) = 0$.

(Many readers may be concerned about the application of calculus to functions involving complex numbers. In particular, a and b in the preceding discussion may be conjugate complex numbers. In that case, $k[(x-a)(x-b)]$ may be written as $k[x^2-(a+b)\cdot x+ab]$; $a + b$ and ab are then real. The first derivative is thus $k[2x-(a+b)]$ as was noted previously.)

Calculus students will recognize that $x = \frac{a+b}{2}$ will yield the maximum or minimum value of f(x). If $k > 0$, a minimum is achieved while if $k < 0$, a maximum is achieved. The value of x yielding the maximum or minimum of f(x) thus occurs at the arithmetic mean of the two roots of $f(x) = 0$. Interpreted graphically, this means that the x-value of the vertex of a parabola is midway between the two x-intercepts, if they exist, of that parabola.

Consider now a cubic polynomial function f(x) with real coefficients. Let a, b, and c be the roots of the equation $f(x) = 0$. Recall that a, b, and c are either all real or consist of one real and two conjugate complex numbers. Find f''(x):

$$f(x) = k[(x-a)(x-b)(x-c)]$$
$$f'(x) = k[(x-a)\frac{d}{dx}[(x-b)(x-c)] + (x-b)(x-c)\frac{d}{dx}(x-a)]$$
$$f'(x) = k[(x-a)[(x-b)\cdot 1+(x-c)\cdot 1] + (x-b)(x-c)\cdot 1]$$
$$f'(x) = k[(x-a)(x-b)+(x-a)(x-c)+(x-b)(x-c)]$$
$$f''(x) = k[[(x-a) + (x-b)] + [(x-a)+(x-c)] + [(x-b)+(x-c)]]$$
$$f''(x) = 2k[(x-a)+(x-b)+(x-c)]$$
$$f''(x) = 2k]3x-(a+b+c)]$$

Solving $f''(x) = 0$, $x = \frac{a+b+c}{3}$. The root of $f''(x) = 0$ is thus the arithmetic mean of the three roots of the equation $f(x) = 0$.

Calculus students will recognize that $x = \frac{a+b+c}{3}$ will be the x-coordinate of the inflection point on the graph $y = f(x)$. Interpreted graphically this means that the x-value of the inflection point of the graph of a cubic equation is the arithmetic mean of the three x-intercepts, if they exist, of the cubic curve.

Challenge for the reader: In the case of a second degree polynomial, the root of the first derivative was the arithmetic mean of the two roots of the original polynomial. In the case of a third degree polynomial, the root of the second derivative was the arithmetic mean of the three roots of the original polynomial. Show that this pattern continues, that is, in the case of a fourth degree polynomial, the root of the third derivative is the arithmetic mean of the four roots of the original polynomial. In general, in the case of an n^{th} degree polynomial, the root of the $(n-1)$st derivative is the arithmetic mean of the n roots of the original polynomial. The proof of this general case is an excellent example of a situation requiring the technique of mathematical induction.

Question for the reader: Are there possible physical interpretations for this process involving polynomials of degree 4 or more?

Curriculum Development and the Basics

by Dr. Marlow Ediger

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The lay public and educators are concerned about emphasizing the basics in the school curriculum. There, of course, is not precise agreement pertaining to which curriculum areas comprise the basics. Lay people and selected educators generally have identified the three R's—reading, writing, and arithmetic—as being represented in the basic curriculum areas. The feeling and thinking exists that pupils need to be competent readers, writers, and computers to become proficient members in society. Recent criticism, in particular, by law citizens and selected educators has been that pupils are not achieving as well as possible in the three R's. Thus, professionals in the school setting need to provide leadership to improve the curriculum in the school-class setting. To improve the curriculum, selected questions and problems need identification and attempts made in obtaining needed solutions.

Problem Solving in the Curriculum

With adequate study, deliberation, and analysis, problematic areas in the school curriculum, such as the following need possible solutions:

1. Which curriculum areas may truly guide pupils to become optimal contributors in school and in society?
2. Within the framework of each basic curriculum area, which understandings, skills, and attitudinal objectives should learners achieve?
3. Which methods of teaching might best aid pupils to achieve these basic learnings, e.g. inductive versus deductive procedures, and stimulus-response versus gestalt schools of thought in terms of how pupils learn?
4. Which philosophy of education may guide each person to achieve optimal development, e.g. John Dewey's experimentalism, realism as exemplified by behaviorists with measurable objectives, individuals basically choosing objectives as advocated by existentialists, or the thinking of idealists as exemplified by universal ideas?
5. Specifically, whose school of thought in psychology should be emphasized in teaching-learning situations, e.g. B.F. Skinner and programmed learning, Jerome Bruner and the structure of knowledge gained inductively by pupils, Jean Piaget and adherence to pupil stages of maturation in teaching-learning situations, and/or Robert Gagné and the hierarchy of objectives?
6. Which plan of grouping pupils for teaching and learning might help each learner to attain optimal development, e.g. nongraded classes, dual progress plans, learning centers and open spaces, schools without walls, or homogeneous versus heterogeneous grouping?
7. How should each basic in the curriculum be organized, e.g. as a separate subject, correlated with another curriculum area, fused with several disciplines, or integrated where subject matter loses its boundaries and borders?
8. How much emphasis should each of the following kinds of objectives receive in teaching-learning situations-understandings, skills, and attitudinal ends?
9. How should pupil achievement ultimately be appraised in terms of finality, e.g. through the use of criterion referenced tests developed by teachers, statewide testing programs within the accountability framework, standardized norm referenced tests, programmed procedures, or through teacher-supervisor observation of learner achievement? What role should pupils have in appraising their own individual achievement?

Concluding Statements

There are numerous issues needing resolving in the school curriculum. Teachers, supervisors, and principals need to identify significant issues and after adequate deliberation adopt related justifiable solutions.

Selected References

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RECOMMENDATIONS for the PREPARATION of high school students for COLLEGE MATHEMATICS COURSES

The Mathematical Association of America
The National Council of Teachers of Mathematics

Recommendations

The Board of Governors of the Mathematical Association of America and the Board of Directors of the National Council of Teachers of Mathematics make the following recommendations:

1. Proficiency in mathemtics cannot be acquired without individual practice, We, therefore, endorse the common practice of making regular assignments to be completed outside of class. We recommend that parents encourage their children to set aside sufficient time each day to complete these assignments and that parents actively support the request of the teachers that homework be turned in. Students should be encouraged to develop good study habits in mathematics courses at all levels and should develop the ability to read mathematics.
2. Homework and drill are very important pedagogical tools used to help the student gain understanding as well as proficiency in the skills of arithmetic and algebra; but students should not be burdened with excessive or meaningless drill. We, therefore, recommed that teachers and authors of textbooks step up their search for interesting problems that provide the opportunity to apply these skills. We realize that this is a difficult task, but we believe that providing problems that reinforce manipulative skills as a byproduct should have high priority, especially those that show that mathematics helps solve problems in the real world.
3. We are aware that teachers must struggle to maintain standards of performance in courses at all levels from kindergarten through college and that serious grade inflation has been observed. An apparent growing trend to reward effort or attendance rather than achievement has been making it increasingly difficult for mathematics teachers to maintain standards. We recommend that mathematics departments review evaluation procedures to insure that grades reflect student achievement. Further, we urge administrators to support teachers in this endeavor.
4. In light of 3 above, we also recognize that advancement of students without appropriate achievement has a detrimental effect on the individual student and on the entire class. We, therefore, recommend that school districts make special provisions to assist students when deficiencies are first noted.
5. We recommend that cumulative evaluations be given throughout each course, as well as at its completion, to all students. We believe that the absence of cumulative evaluation promotes short-term learning. We strongly oppose the practice of exempting students from evaluations.
6. We recommend that computers and hand calculators be used in imaginative ways to reinforce learning and to motivate the student as proficiency in mathematics is gained. Calculators should be used to supplement rather than to supplant the study of necessary computational skills.
7. We recommend that colleges and universities administer placement examinations in mathematics prior to final registration to aid students in selecting appropriate college courses.
8. We encourage the continuation or initiation of joint meetings of college and secondary school mathematics instructors and counselors in order to improve communication concerning mathematics prerequisites for careers, preparation of students for collegiate mathematics courses, joint curriculum coordination, remedial programs in schools and colleges, an exchange of successful instructional strategies, planning of in-service programs, and other related topics.
9. Schools should frequently review their mathematics curricula to see that they meet the needs of their students in preparing them for college mathematics. School districts that have not conducted a curriculum analysis recently should do so now, primarily to identify topics in the curriculum which could be either omitted or de-emphasized, if necessary, in order to provide sufficient time for the topics included in the above statement. We suggest that, for example, the following could be de-emphasized or omitted if now in the curriculum:
   (a) Logarithmic calculations that can better be handled by calculators or computers,
   (b) extensive solving of triangles in trigonometry,
   (c) proofs of superfluous or trivial theorems in Geometry.
10. We recommend that algebraic concepts and skills be incorporated wherever possible into geometry and other courses beyond algebra to help students retain these concepts and skills.

A Vector Demonstration That the Circumcenter, Orthocenter, and Centroid of a Triangle Are Collinear

by Katherine Woerner

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Austin, Texas

The proof which is presented herein is delightful in that it is within the capability of most students in a high school analysis class who have just completed a discussion concerning vectors. The proof involves many concepts from geometry, algebra, analytical geometry, and vector theory familiar to the student.

The main concepts used herein are

(1) The midpoint formula
(2) The addition of vectors
(3) The product of a scalar and a vector
(4) The norm of a vector
(5) The inner product of two vectors
(6) Two vectors are normal (perpendicular) if and only if their inner product is zero
(7) Two vectors are collinear if and only if one is a scalar multiple of the other and they have the same direction if and only if the scalar is a positive number
(8) The norm of the vector sV where s is a scalar and V is a vector is the product of |x| times the norm of V

The development assumes the existence of the circumcenter, centroid, and orthocenter of a triangle as defined by Adler (1).

PROBLEM: Suppose A, D, and C are the vertices of a triangle and a rectangular coordinate system is imposed on triangle ABC so that the coordinates of A, B, and C are (b,c), (0,0), and (a,0), respectively, where a ≠ 0 and c ≠ 0.

Use vector theory to

(1) Find the coordinates of the circumcenter, centroid, and orthocenter of triangle ABC.
(2) Show that the circumcenter, centroid, and orthocenter are collinear.

Two other interesting results which can easily be shown true are

(1) The centroid divides the line segment from the orthocenter to the circumcenter in a ratio of 2:1.
(2) The centroid divides the medians of the triangle in a ratio of 2:1 from a vertex to the midpoint of the opposite side of the triangle.

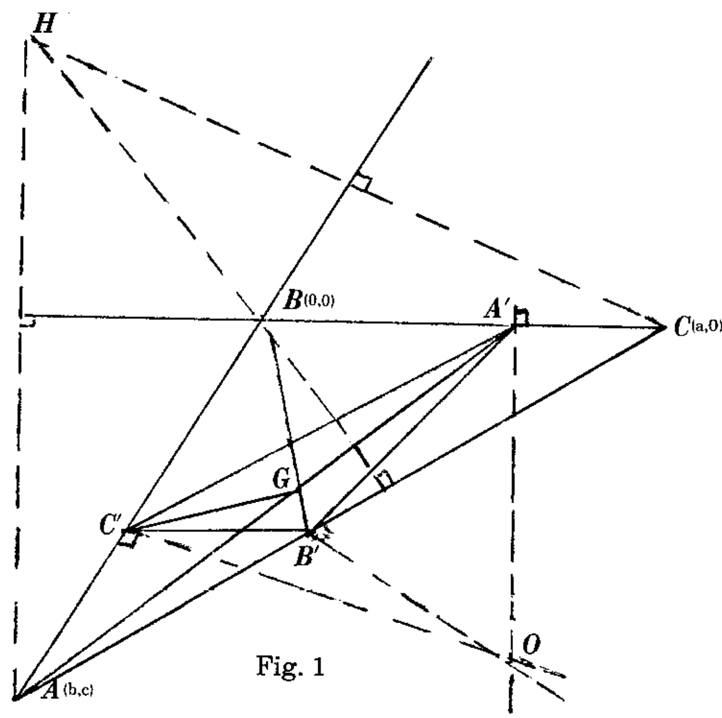

Fig. 1

SOLUTION: Suppose A, B, and C are the vertices of a triangle and a rectangular coordinate system is imposed on triangle ABC so that the coordinates of A, B, and C are (b,c), (0,0) and (a,0), respectively, where a≠0 and c≠0.

Let A', B', and C' denote the midpoints of BC, AC, and AB, respectively. Now the coordinates of A' are (a/2,0), the coordinates of B' are ((a+b)/2,c/2), and the coordinates of C' are (b/2,c/2).

Use the notation $\overrightarrow{PQ}$ to denote the vector from point P to point Q.

Since the circumcenter 0 of the triangle ABC is the point common to all the perpendicular bisector of the sides of triangle ABC, A'0, B'0, and C'0 are normal to $\overrightarrow{BC}$, $\overrightarrow{AC}$, and $\overrightarrow{BA}$, respectively. (See figure 2.) Therefore, the inner products $\overrightarrow{A'0}\cdot\overrightarrow{BC}$, $\overrightarrow{B'0}\cdot\overrightarrow{AC}$, and $\overrightarrow{C'0}\cdot\overrightarrow{BA}$ are all zero.

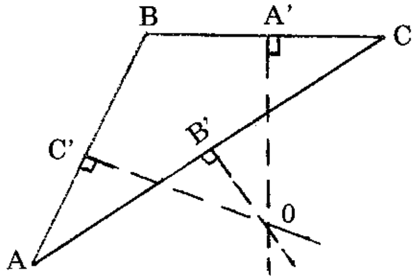

Fig. 2

Let (x,y) denote the coordinates of 0.

Since $\overrightarrow{A'0}\cdot\overrightarrow{BC} = (x-a/2,y)\cdot(a,0)$
$= ax-a^2/2$
$= 0,$

it follows that $ax = a^2/2$

and $x = a/2$ since $a\neq 0$. This is an expected result because the line A'0 is a vertical line through A':(a/2,0).

Also, $\overrightarrow{B'0}\cdot\overrightarrow{AC} = (x-(a+b)/2, y-c/2)\cdot(a-b,-c)$
$= ax-(a^2+ab)/2-bx+(ab+b^2)/2-cy+c^2/2$
$= (a-b)x-cy+(-a^2+b^2+c^2)/2$
$= 0.$

Substituting a/2 for x into the above statement,
$$(a-b)(a/2)-cy+(-a^2+b^2+c^2)/2 = 0$$
which simplifies to
$$-cy+(b^2+c^2-ab)/2 = 0$$
and solving for y yields
$$y = (b^2+c^2-ab)/2c, \text{ since } c\neq 0.$$

Testing shows that 0:$(a/2,(b^2+c^2-ab)/2c)$ satisfies the definition of the circumcenter of triangle ABC.

Since the centroid G of triangle ABC is the point common to the medians of triangle ABC,
$$\overrightarrow{BG} = k_1(\overrightarrow{BB'})$$
and $\overrightarrow{BG} = \overrightarrow{BC} + k_2(\overrightarrow{CC'})$, where each of k_1 and k_2 is a real number which will be determined. (See figure 3.)

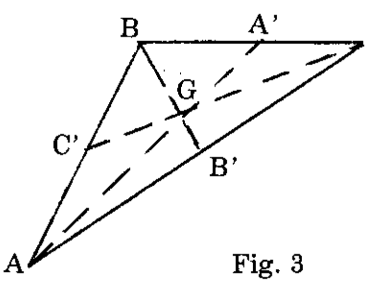

Fig. 3

Let (x,y) denote the coordinates of G.

Now, $\overrightarrow{BG} = k_1(\overrightarrow{BB'})$

or $(x,y) = k_1((a+b)/2,c/2)$

and it follows that

(1) $x = k_1((a+b)/2)$ and $y = k_1(c/2)$.

Also, $\overrightarrow{BG} = \overrightarrow{BC} + k_2(\overrightarrow{CC'})$

or $(x,y) = (a,0) + k_2((b-2a)/2,c/2)$

which results in

(2) $x = a + k_2((b-2a)/2)$ and $y = k_2(c/2)$.

Since $y = k_1(c/2)$ and $y = k_2(c/2)$, it follows that $k_1 = k_2$,

and substituting k_1 for k_2 into the expression for x in (2) and using (1) yields
$x = k_1((a+b)/2$
$= a + k_1((b-2a)/2).$

Rearranging, $k_1((a+b)/2 - (b-2a)/2) = a$

or $k_1(3a/2) = a$

and $k_1 = 2/3$, since $a\neq 0$.

Using 2/3 for k_1 in (1) gives the results that
$$x = (a+b)/3 \text{ and } y = c/3.$$

Testing shows that G:$((a+b)/3,c/3)$ satisfies the definition of the centroid of triangle ABC.

Since the orthocenter H of triangle ABC is the point common to all the altitudes of triangle ABC, $\overrightarrow{AH}$, $\overrightarrow{BH}$, and $\overrightarrow{CH}$ are normal to $\overrightarrow{BC}$, $\overrightarrow{AC}$, and $\overrightarrow{AB}$, respectively. (See figure 4.) Hence, the inner products AH·BC, BH·AC, and CH·AB are all zero.

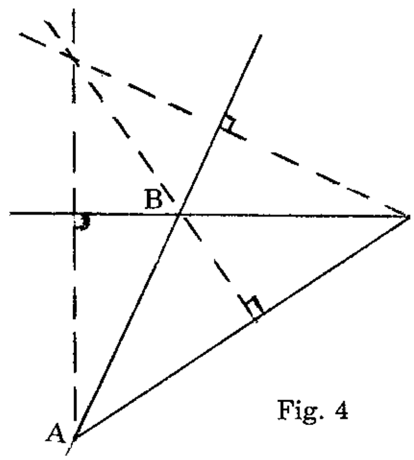

Fig. 4

Let (x,y) denote the coordinates of H.

Since $\overrightarrow{AH}\cdot\overrightarrow{BC} = (x-b,y-c)\cdot(a,0)$
$= ax - ab$
$= 0,$

it follows that $ax = ab$

and $x = b$, since $a\neq 0$.

Also, $\overrightarrow{BH}\cdot\overrightarrow{AC} = (x,y)\cdot(a-b,-c)$
$= xa-xb-yc$
$= 0.$

Substituting b for x into the above statement,
$$ba-b^2-yc = 0$$
and solving for y yields,
$$y = (ab-b^2)/c, \text{ since } c\neq 0.$$

Testing shows that H:$(b,(ab-b^2)/c)$ satisfies the definition of the orthocenter of triangle ABC.

Now H:$(b,(ab-b^2)/c)$, G:$((a+b)/3,c/3)$, and 0:$(a/2, (b^2+c^2+ab)/2c)$ are collinear if $\overrightarrow{HG}$ is the product of a scalar times $\overrightarrow{G0}$.

Consider $\overrightarrow{HG} = ((a+b)/3-b, c/3-(ab-b^2)/c)$
$= ((a+b-3b)/3,(c^2-3ab+3b^2)/3c)$
$= 2((a-2b)/6,(c^2-3ab+eb^2)/6c)$
$= 2(a/2-(a+b)/3,(c^2+b^2-ab)/2c-c/3)$

$= 2\overrightarrow{G0}$.

Therefore, $\overrightarrow{HG}$ and $\overrightarrow{G0}$ are collinear and the points H, G, and 0 are collinear.

The above argument also shows that centroid G divides the line segment from the orthocenter H to the circumcenter 0 in a ratio of 2:1.

Note that
$$\overrightarrow{BA} + 2/3(\overrightarrow{AA'}) = (b,c)+2/3((a-2b)/2,-c) = (b+(a-2b)/3,c-2c/3) = ((a+b)/3,c/3) = \overrightarrow{BG}.$$

This argument and the previous argument concerning centroid G proves that the centroid divides the medians of triangle ABC in a ratio of 2:1 from a vertex to the midpoint of the opposite side of the triangle.

This completes the proof.

References

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