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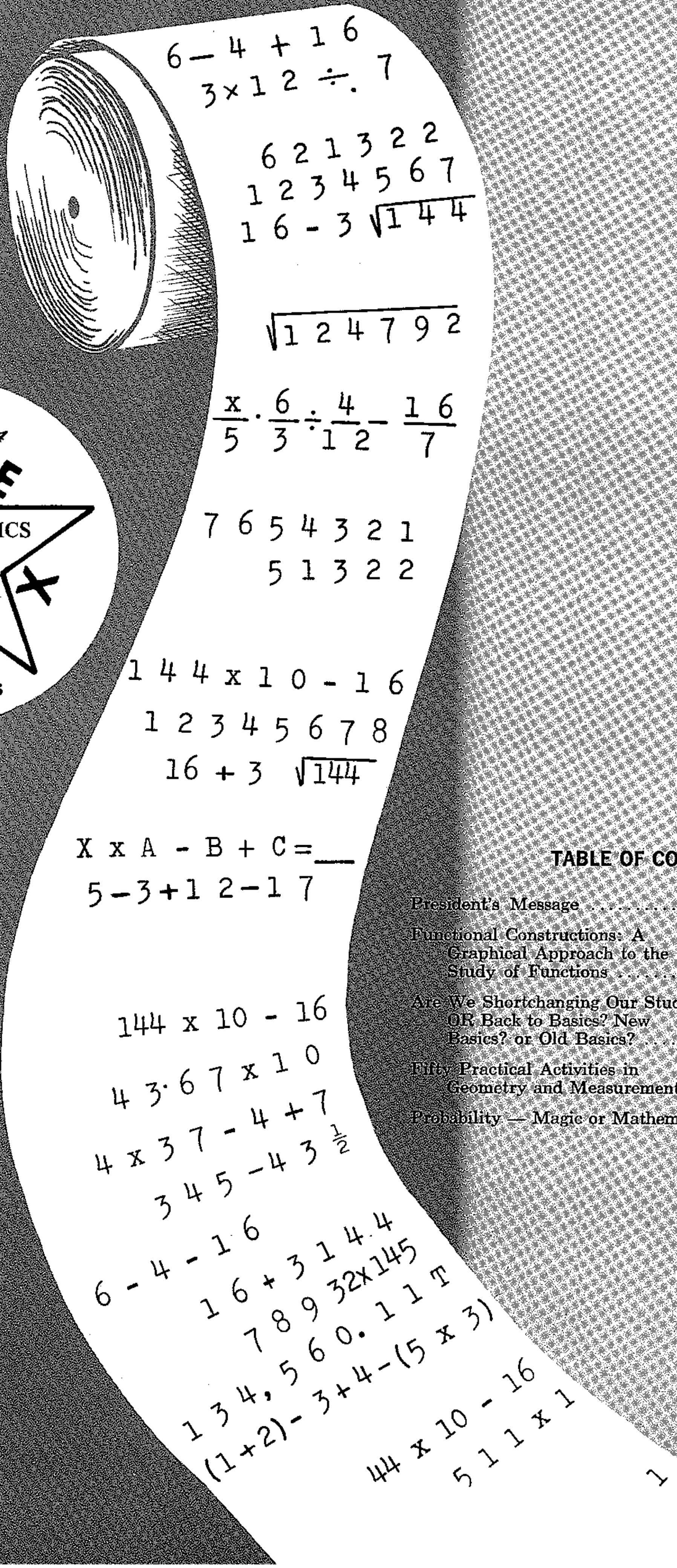

TABLE OF CONTENTS

TEXAS MATHEMATICS TEACHER is the official journal of the Texas Council of Teachers of Mathematics. The views expressed are the contributor's own and are not necessarily those of the publisher or the editor. All manuscripts and correspondence about this publication should be addressed to Mr. J. William Brown, *Texas Mathematics Teacher,* 100 So. Glasgow Drive, Dallas, Texas 75214.

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President's Message

With the many adoptions of texts to be made this year, along with the workshops, mathematics tournaments, conventions, and the usual work teachers encounter this year has all the appearances of being a busy one for all of us. Most of us accept these things as part of our profession and accept them graciously. I hope all of us will take an extra step toward professionalism and become involved in mathematics education through the local councils, our own organization, and NCTM. This is not a request for you to renew your membership, although many of you need to do so, but more importantly to become actively involved with other mathematics teachers to promote mathematics curriculum improvement, improve teacher competencies, and arouse student interest throughout the state.

I would like to pose a question—Is it possible to have a competent teacher, using a relevant curriculum, and meeting his or her class everyday with viable enthusiasm for mathematics and have no change in student interest and performance? Surely, in such a setting as this, something must happen in student learning.

With this message, and at the close of CAMT, I yield the presidency to Shirley Cousins. It has been an enjoyable, though sometimes frustrating, two years. Much has been done toward improving our status, yet there is much to be done. To those of you who offered encouraging words and to those of you who offered suggestions for improvement, I express my deepest appreciation.

To my principal, supervisor, and superintendent who granted me time to work and attend meetings, I am also deeply indebted.

But most of all, to each of you, I am appreciative of the opportunity you have given me to serve. I am certain you will find, with Shirley as president, that even though we have had a good past the best is yet to come.

BILL ASHWORTH

Functional Constructions A Graphical Approach to the Study of Functions

By Madolyn J. Reed
Assistant Director of Mathematics
Houston Independent School District

Most researchers seem to agree that some students have difficulty with purely abstract concepts. As a result, a wealth of manipulative devices for classroom use has been developed to provide concrete experiences for these students. This development of materials for better communication has in some cases left the mathematics teacher, well-trained in the art of abstracting, at a disadvantage in finding good mathematical concepts easily adaptable to presentations with manipulative objects. It is as if one is given a telephone without being aware of the use of the device for communicating with someone specifically. One might very easily discover that several spins of the dial will cause a connection to be made with someone, somewhere, but how does one reach Aunt Bessie in Prairie View, Texas? Surely there is more to be done with this instrument than random dialing. So it is with the straightedge and compass in mathematics.

The function concept is an important one, worthy of several chapters' consideration in most senior high school mathematics texts and, in some cases, worthy of consideration as a separate course. While this "show me" type student might have difficulty with a completely abstract approach to the study of functions, he might very well feel competent with a combination of abstract and concrete considerations in the study of the topic.

By the time a student enrolls in second year algebra, he has had adequate exposure to both set terminology and set notation. Customarily, a function is defined as a relation, i.e., a set of ordered pairs, in which no two pairs have the same first coordinate. Symbolically, if F is a function, $F = \{(a,b)\colon a \varepsilon D$ and $b \varepsilon R$ where D and R are sets.$\}$. Furthermore, if $(a,b) \varepsilon F$ and $(a,c) \varepsilon F$ then $b = c$.

Regardless of the approach, the definition almost always uses the word set or set notation. Now then, a set is usually specified in one of three ways, by rule, by roster or by graph. It is the latter method of specification that the writer has concentrated on in an effort to communicate to students the concept in general and the other methods of specification in particular. This is in keeping with the recommendation that a "from concrete to abstract" approach be taken.

Presupposing a background in simple graphing, the technique of plotting a point in a conventional rectangular coordinate system, the straightedge and compass technique for finding the sum and difference to two directed line segments, we begin by a consideration of the *graphical* definition of a function by having the student plot points in an effort to determine what happens to the graph of two ordered pairs with the same first coordinate. In most cases the response will be immediately obvious. The graphs of two ordered pairs with the same first coordinate will lie on the same vertical line. In less sophisticated language, "The two points are one over the other." At this point a reference to the algebraic definition of a function should be given and a discussion of what this means graphically follows. The transition should be smooth and thoroughly understood. Given a simple relation and a method of picturing the relation, the student can decide if the relation is a function by looking at the graph. The relation is a function if no vertical line intersects the graph in more than one point. Later in the course, a more sophisticated analysis of whether or not a given relation is a function can be considered.

It is customary to next consider the meaning of domain and range of a function. The exercise here involves plotting points and establishing the relationship between the coordinates of the points and the coordinates of the projections of the points on each axis. It is important that the idea of the coordinate of the point, which is the projection of the graph of the ordered pair on the horizontal axis, is the first coordinate of the ordered pair; a similar projection onto the vertical axis will give the second coordinate of the point. In addition, certain line segments are important to this development. The directed distance from the vertical axis to the point is also the first coordinate of the point, while the directed distance from the horizontal axis is the second coordinate. This second segment is the one with which we will be greatly concerned.

Exercises involving projections of figures on each axis are advisable here so that these students can intuitively *see* the relationship between a graph and its projection as introductory to the definition of domain and range of a function. Again it is time now to read the algebraic definition of domain and range to reconcile the definition with what is being done. The domain of a relation, i.e., the set of all first coordinates is the projection of the graph onto the horizontal axis, and the range of the relation is its projection onto the vertical axis.

These first two concepts, "Is it a function?" and "What is the domain and range?" are usually as much as most students really understand in a completely algebraic presentation. It is at this point in the development then, that he needs something to keep him interested without sacrificing the understanding of what is being done.

Now we present the arithmetic of functions. Of importance is the graphical representation of the value of function f at x where x belongs to the domain of f. Recall that this means that for any x, a point on the horizontal axis, $f(x)$ is the second coordinate of the point on the graph whose first coordinate is x. This we have seen can be represented by a line segment. To construct $f(x)$ we erect a perpendicular to the horizontal axis at the point whose coordinate is x. The intersection of this perpendicular and the graph of f (emphasize that this intersection is a point) together with the point on the horizontal axis whose coordinate is x determines, i.e., singles out, a unique line segment. The length of this segment is $f(x)$.

Definition: If f is a function and g is a function then the function h, whose domain, D, is the intersection of the domains f and g, and in which $h(x)$ is $f(x) + g(x)$ for all x in D is the sum of f and g. Symbolically $h = f + g$. We resort to graphs again to avoid mass confusion. To construct the sum of two functions, f and g, we select a point, z, in the intersection of the domains of the two functions. At the point z a perpendicular is constructed to the horizontal axis. On this perpendicular segment having one of its end points at z and equal in length to $f(z) + g(z)$ is constructed. Recall that we presupposed a knowledge of such a construction at the beginning of the development. The point which is the upper end point of the segment so constructed is a point in h. Furthermore the length of the segment itself is $h(x)$. The function $g(x) = 2x - 2$ can be drawn with the segments $g(x)$, $x \varepsilon \{-5, -4, -3, \ldots 3, 4\}$ represented. A second function $f(x) = x^2$ with the segments $f(x)$, $x \varepsilon \{-3, -2, \ldots, 3\}$ can be represented. The sum of the two functions $f(x) + g(x)$ or $h(x) = x^2 + 2x - 2$ may be shown with the segments $h(x)$, $x \varepsilon \{-5, -4, -3 \ldots 2, 3\}$ represented.

The problem of the difference of two functions is a simple one once the class understands the simple construction of finding the negative of a directed line segment. With this understanding and a refresher comment about the definition of subtraction in the set of real numbers, i.e., $(a - b)$ is $[a + (-b)]$ the student should be encouraged to develop the construction for $f - g$ and $g - f$. Here is a magnificent opportunity to have the student see another illustration in which subtraction is not a commutative operation.

Before we can construct the product of two functions, we must look at the construction of the product of two segments. This construction is usually not one which has been done by these students. The construction presented to them requires the establishment of a unit segment. This is not difficult to do since constructions were done in a coordinate system. The teacher should illustrate the product construction of two segments a and b, with different possibilities for a and b. The results are interesting when one of the segments is 0 or when one segment is 1. The construction, then, is to locate the point whose coordinate is b on the horizotnal axis. A discussion of the commutativity of multiplication is in order here. At the point on the horizontal axis whose coordinate is $b - 1$, construct a perpendicular of length a. The upper end of this perpendicular and the point b determine a line which intersects the vertical axis at a point whose coordinate is the product of a and b.

Once the nature of the construction is understood, the application to constructing values in the product of two functions naturally follows. In the presentation of the product of two functions some new considerations for discussion arise. Given an x in the intersection of the domains of two functions f and g, segments $f(x)$ and $g(x)$ are easily located now. But a review of the product construction requires that one of the factors $f(x)$ or $g(x)$, be taken along the horizontal axis. Question! What shall we do? The ensuing discussion takes patience on the part of the teacher so that the class has ample time to make a decision. Complete withdrawal from the discussion by leaving the room might be a good idea for it is important that the class sees what must happen here. The idea is necessary for the study of composition of functions and the inverse of a function.

To continue, to construct the product of $f(x)$ and $g(x)$ for any x, select one of the factors, say $f(x)$ as the factor to be constructed on the horizontal axis. With the unit length determined by the system as radius, use the compass to locate $[f(x) - 1]$. At the point $[f(x) - 1]$ construct a perpendicular to the horizontal axis of length $g(x)$. Continuing the construction as indicated previously the segment whose length is $f(x) \cdot g(x)$ is now determined on the vertical axis. The student should be aware that the construction is not complete. The segment we just constructed is $h(x) = f(x) \cdot g(x)$ and must be transferred to the location x on the horizontal axis, i.e., on the line perpendicular to the horizontal axis at x, h (x) must be constructed. Illustrate the product of two linear functions. Now the students can "see" the quadratic function which is evolved as the product of two linear functions.

The construction for the difference of two functions was left to the student as a natural consequence of the definition of subtraction and the construction for finding additive inverses. Similarly the construction for the quotient of two functions f and g with g not constantly zero can be left to the student as an equally natural consequence of the definition of division $(a/b = a \cdot 1/b)$ and the following construction for finding reciprocals.

The construction for finding reciprocals is not difficult but requires knowledge of the construction of a parallel through a given point. A review of this construction may be necessary. Illustrate the reciprocal construction with $a > 0$. On the horizontal axis, locate the point whose coordinate is a. The line through this point and the point on the vertical axis whose coordinate is 1 is drawn. Now through the point whose coordinate is 1 on the horizontal axis construct a line parallel to the first line. This line intersects the vertical axis in a point whose coordinate is the reciprocal of a. Illustrate the same construction with $a < 0$.

An interesting question may be posed is what happens in the construction if $a = 0$? The discussion of this is good reinforcement of the fact that zero has no reciprocal and hence division by 0 is not possible.

By now the student has had ample experience with constructions and will not be hesitant to discuss the composition of two functions f and g. Taking a look at the definition, the composition of two functions f and g, symbolically $f \, 0 \, g$, is defined to be $f[g(x)]$ where $g(x)$ is in the domain of f. Suggest this construction and show the result of constructing $f[g(x)]$ where $f(x) = x$ and $g(x) = \frac{1}{2}x + 1$. The construction this time utilizes the properties of the line with equation $y = x$. (Note: This line could have been utilized in the product construction since any projection onto this line makes it possible to locate a point on either axis having the same coordinate as a point on the other axis.) Locate x, on the horizontal axis, determine $g(x)$. Project the upper endpoint of $g(x)$, horizontally onto the line $y = x$. What is the result of this projection? From this point on $y = x$ project vertically onto f. The distance from this point on f to the horizontal axis is $f[g(x)]$. It only remains to project horizontally again onto the line through x perpendicular to the horizontal axis to complete construction.

The final consideration now is to the concept of the inverse of a function. Definition: $f^{-1} = (a,b)$ if and only if $f = (b,a)$. We need to establish a relationship between the point in the plane whose coordinate is (a,b) and the point in the plane whose coordinate (b,a). To encourage this discovery by the class, have the class plot pairs of points whose coordinates are related by this property. All line segments whose lengths can be determined as a result of plotting pairs of points (a,b) and (b,a) should be drawn in. Show the result of this exercise for selected pairs.

A discussion of symmetry is needed here. Draw in the line whose question is $y = x$. Illustrate that the point whose coordinate is (a,b) and the point whose coordinate is (b,a) are the ends of a segment whose perpendicular bisector is the line whose equation is $y = x$. This is precisely what we mean when we say the points are symmetric to each other with respect to the line whose equation is $y = x$. If the revolving of the graph about the line whose equation is $y = x$ is not readily suggested, it might be prompted by having the student complete one construction with the graph of f one color and the graph of f^{-1} another. Then have him revolve his paper 180° about the line whose equation is $y = x$ and view the graph from the back side of his paper. Then have him complete the same exercise with the graph of f only and compare the view from the back side of his paper with the graph he constructed of f^{-1}. Is f 1—1? It is if f^{-1} is a function. The student should be able to describe a test to determine the answer by looking at the graph.

The set of all functions with the algebra defined as it is in this paper makes an interesting study of a system in which the following questions can be asked: Are the operations commutative? Is this set a field? Are the operations associative? Distributive? These questions can be answered with a graphical presentation. It is not intended that these concepts be presented in isolation but should be accompanied by an algebraic presentation using the algebra as a language to describe the results of the experiments. There are numerous opportunities for tangent chasing into other algebraic concepts via constructions from geometry.

Are We Shortchanging Our Students? or Back to Basics? New Basics? or Old Basics?

By Frank Ebos
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University of Toronto.

Probably each of us in our role as a teacher has been confronted by an anxious student inquiring about some of the math we teach. How many times have each of us heard:

Why are we taking this stuff?

What's it good for . . . . sir?

I would be the first to admit that we can't justify everything we do *all the time.* To explain how a specific topic in mathematics "fits" into the scheme of things is often difficult. The student must trust that what we are doing day by day in the math classroom is useful, is relevant, and is needed for today's activities as well as for tomorrow's. Unfortunately, for many students tomorrow never comes. The students then become parents, and the cycle of asking "Why are we taking this stuff" is continued.

There are certainly many indications that there is concern about the curricula we teach. Many studies have been, and are being completed, in both the United States and Canada to ascertain "What should be the math curricula?" Unfortunately, to predict the content, the skills, or even the methods needed at some future date is difficult. If you listen to the experts and read the journals, you soon would develop a complex about what we are not doing in the math classroom, but thank goodness for the so-called new math. We have in it at least a scape-goat. We have seen the headlines "New Math has failed! Back to basics!", and now we have a replacement for the new math, the BTB (Back To Basics). When the "New Math" was in Vogue, each person you talked to had a different "understanding" of what the "New Math" was: There was the *Set-New-Math* followers; the *Base-New-Math* followers; the *Structure-New-Math* followers, etc. (I apologize to those new-math groups not identified at this time). The parents identified new math from another point of view. They only saw what their children brought home; sets, and new words: commutative, associative, distributive, inverse, base 2, base 8, third base — to name a few — and soon the scene is set for the BTB to be formed, and the parents are willing to join.

These days, I am hearing more and more of the BTB, but I think that before we change we need to decide *"what are the basics"*. Parents whose experience with math was almost completely computation, evaluate a new program or curriculum according to their backgrounds. We must ask: are the basics solely related to computation? I think not. No one would disagree that "the old basics" are essential, but there are 'new basics" that parents as adults use in their every day living, and which need to be dealt with. The computation content, so familiar to many and part of the "Happy Days Syndrome" to return to basics is not, by itself, suitable even for today's world, let alone tomorrow's.

My main concern is not that the BTB wants change (improvement, or whatever) but that they appear to want the pendulum to swing back to a "shut-up: do-this" curriculum, based on computation[1]. We can't afford to have the pendulum swing back too far. Everyone will basically agree that computational skills are important *and* basic to the students mathematical development, but to stop there would short-change our students. Before the BTB make the same errors as the New Math groups, they (there are probably more than one) should decide what the basics are!

I want to help the BTB by offering the following suggestions. (These suggestions could be added to the computational platform already advocated by the BTB.) A little review is useful here:

BTB = a group of persons who want to return *B*ack-*T*o-*B*asics)

1. Students want us to be accountable. We should have some reasonable explanation or justification for "Why are we taking this stuff?" If we were to provide examples of ap-

plications for the present curriculum, some students and parents would be partially satisfied. To tell a student he is taking a topic because he needs it for next year's math is a weak argument. Many topics have applications not only to satisfy the present questions asked by students but also to provide a foundation for their future study of the topic in next year's math class. *Applications should be basic to any curriculum.*

2. In our everyday living, we are never given "neat problems" solved in a "neat way" that result in a "neat solution". A real-life problem requires us to sort out the useful (needed) information from the extraneous information. Do we ever give students problems that contain more information than is really required to solve the problem? Will our students ever be given a problem to solve when they enter the "world of work" for which the "boss" provides only the information that is necessary? I doubt it! You doubt it! Yet how much of this practice in solving do we provide our students in our math classrooms (if you do provide these types of problems, then consider this section a brief review). *To decide on what information is necessary or needed to solve a problem, to me, is a basic skill.* How many times have you given a problem which has a missing piece of information that the student is to provide in order to solve the problem? (Try assigning 5 problems, each having an essential piece of information missing. Provide a second sheet that contains the 5 missing "pieces", as well as 25 other pieces of extraneous but closely related information). This skill will probably be used more often than the skills advocated by the BTB. Without this skill the BTB skills are often confusingly applied by students to solve problems.

   However, "let me make it perfectly clear" that we need the BTB basic skills once the information is properly interpreted and the essential computational operations decided upon. Unfortunately, "good old" Euclidean geometry provided an opportunity for a student to "sort out" the needed information, but an EGID movement (Euclidean Geometry Is Dying) seems to have sprung up (E GAD!)

3. Every day, as adults, we read the Gallup Poll, the latest statistics on "why we are paid more and more but are eating less and less" and, this poll and that poll. However, do we provide any basics for students to tackle the world of statistics? We give a brief look at the topic in the seventh and eighth grade (if at all in some classrooms) and then almost completely ignore it in the ninth and tenth grade, but give some hope to those who stay on and finish high school (the key question here is—how many will finish high school?) Could these students not find the skill of working with statistics useful just for everyday living and thus be given the opportunity to study the topic in the ninth grade. *Working with, interpreting, and reading statistics is a basic skill.*

   Statistics permeate too much of our everyday living to be ignored as it has been in our mathematics curriculum. We are short changing our students by not teaching this skill.

4. How many students have asked you "Is this right?". How many are completely lost if they cannot find the answer at the back of the book? Students should be taught (and taught and taught . . . .) to know how to check that their answers are reasonable. *To know the answer is reasonable is a basic skill.* A student must by the twelfth grade "feel" whether the answer is reasonable. The "boss" does not assign problems with the answer at the back of the book. Teaching this skill partially can be accomplished by providing students with skills of approximating, estimating, as well as decision making. Too often students are given problems that have "neat" answers. The following problem about a corn roast seems trivial at first but when it was assigned to eighth grade students it introduced the need to make decisions.

   "How much would it cost our class to go on a corn roast?"

   The students list the assumptions, and make decisions to arrive at the cost of going on the corn roast. I have assigned this problem to teacher groups and have had to referee arguments, as well as to impose a maximum time limit of fifteen minutes because the problem, although simple in appearance, can be complex in finding a "reasonable" solution. The solution will eventually involve answers to these questions:

   Where are we going? How do we get there? Do we take drinks? Do we need salt, butter, napkins, pepper, etc. etc.? These are but a few of the problems hidden in arriving at a "reasonable" price for a "reasonable" corn roast. What's your answer?

   Problems suitable for different grade levels should be assigned to develop and strengthen this basic skill "Is my answer reasonable?"

5. I am sure many teachers already provide students with a variety of strategies for "starting" to solve a problem. I will never forget when I posed a problem to a class and got the immediate reply, "We haven't taken that yet . . . . sir!" We all remember too well the complete blanks left for some problems tackled by students writing exams. Some students freeze as soon as we say "word problem", and will sit and look but really do nothing to "start" the problem.

Teaching students to "sketch" the problem or make a diagram to help them solve the problem is a basic skill that needs to be emphasized continually. Many problems have been solved by "doodlers". Often a problem is solving the "doing" something rather than "waiting" for an inspiration. A blank page provides very little inspiration (but there are exceptions, of course).

My list of recommendations to the BTB is not exhaustive or complete. Each of us have probably many other recommendations to add, but *we do* have to make recommendations.

There is much going on in curriculum development as well as studying how students learn. The process is painstakingly slow. Studies are being conducted on developing the problem-solving abilities of our students, as well as on many other aspects of teaching and learning.

Curriculum development is progressing, but to a new teacher the educational scene must appear confusing. I once heard a description of curriculum development that seems to describe the present scene. The scene opens in the cockpit of a 747 (big bird). One pilot remarks to the other "We seem to be in a fog". The other pilot remarked, "Yes, but *we are* making headway." The daily evidence, as valid or invalid as we wish to make it, seems to indicate that there are some shortcomings in our curriculum, and different pressure groups are making it known that our present curriculum is *not* making headway. Students do not want to be short-changed, nor do we want to short-change them. They want to be able to understand "why we are taking this stuff"; and perhaps their reoccurring question might indicate a weakness in our curriculum. We are so over-preparing for the future that we are neglecting the present. We do have to have informed consumers. We want them to use math in their everyday living (accurately too!) We do want our students to be mathematically-literate when they read the newspaper, pay that charge account or calculate the percentage increase of their raise.

At the same time, we do not want to short-change the students by providing them with a curriculum that accommodates the present but ignores the future. They must have some "computer sense", some appreciation and working knowledge of calculators, as well as an appreciation of mathematics, "as a science". At the same time, we need to prepare them to be mathematically sound, and for this we would need to provide a useful mathematical foundation.

To keep everybody happy is impossible, but we all have to participate in lifting the fog.

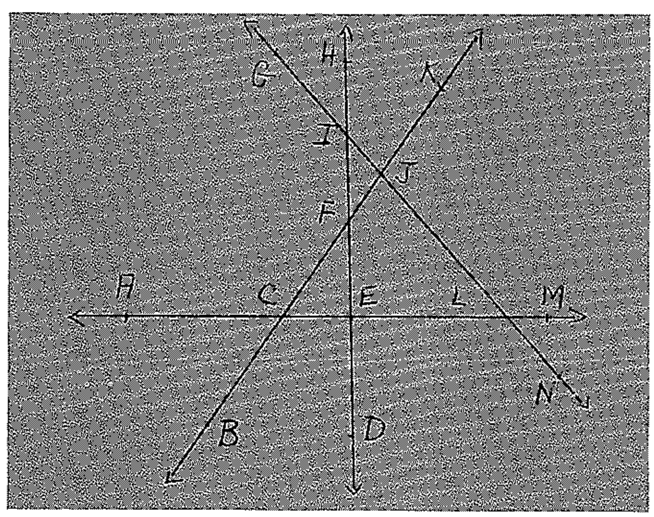
1. A description of the "shut-up: do this" curriculum was in a speech delivered by Eric MacPherson of the Faculty of Education, University of Manitoba at the Annual Meeting of the National Council of Teachers of Mathematics in Denver, Colorado, April, 1975.

Fifty Practical Activities in Geometry and Measurement

By James R. Smart
San Jose State University, California

Now that many teachers are beginning to want laboratory-type activities or projects for use in their mathematics classes, the available supply of practical and interesting suggestions has proved much shorter than expected. The following list of projects in geometry and measurement hopefully includes some that are new and worthwhile for you to try. To make the list of more value, four possible symbols have been added after each activity. Those marked with a C can be used as applications of a hand-held calculator. Those marked with an O are suggestions for activities outside the ordinary classroom. Those marked with an S probably require some high school mathematics. Projects marked with an M use metric units.

Figure 1



Activities Involving Sets of Points:

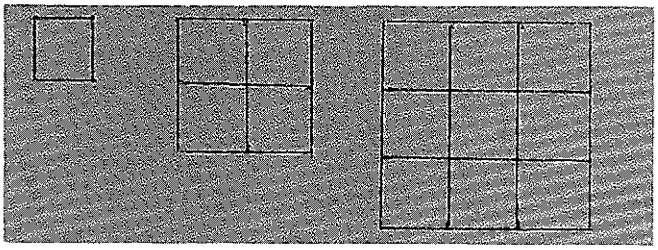
1. Play a treasure hunt game. Example: The treasure is buried at one of the labeled points in Figure 1. Use these three clues to locate it.

a. Not on $\overleftrightarrow{DH}$. b. On I side of $\overleftrightarrow{AL}$. c. Not on $\overleftrightarrow{GN}$.

2 Make up a treasure hunt game and let a friend find the treasure.

3. Find the pattern for the total number of squares of all sizes in arrays of squares such as those in Figure 2, without counting each square. (C)

Figure 2

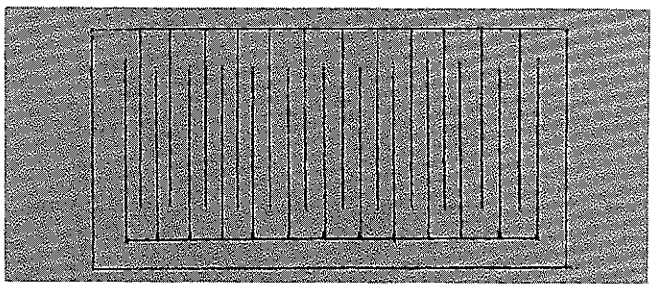


4. Find a formula for the total number of diagonals in a polygon with any given number of sides. (C,S)

Activities Involving Length

5. In a sheet of paper, cut a hole big enough to walk through. One pattern is given in Figure 3.

Figure 3

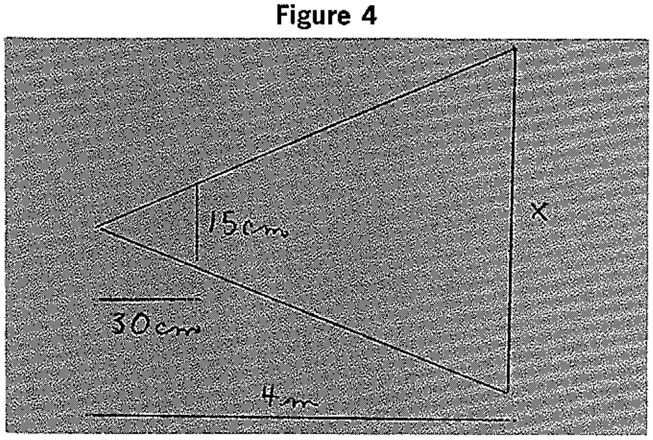


6. Use a state map. Make a marker for the scale in kilometers and practice finding distances between cities. (M)

7. Practice estimating heights of objects by comparison of shadows. For example, if the shadow is 3m long for an object known to be 2m tall, then an object with a shadow 6m long will be 4m tall. (C,O,M)

8. Use a method of indirect measurement with a ruler to estimate heights. As shown in the example in Figure 4, sight along a centimeter ruler, then use similar triangles and the proportion $x/4 = 15/30$, so that $x = 2(\text{m})$. (C,O,M)

Figure 4



9. To help estimate with centimeters, construct a set of length standards from cardboard in multiples of 4 centimeters. Then use these standards to practice estimating. (M)

10. Choose an appropriate scale and make scaled drawings in metric units for a room and a table. (C,M)

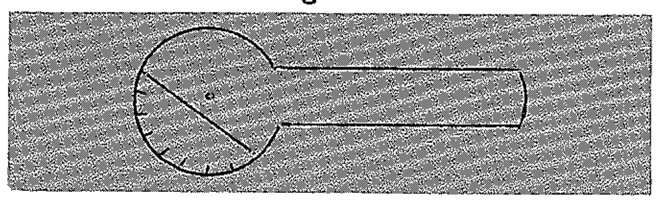
11. Practice using a micrometer to measure shorter distances such as the thickness of a sheet of paper. (M)

12. Review non-standardized body units of length such as the cubit, span, palm and hand, and use these units to estimate lengths of objects.

13. Using a local map, list some of the features located within a radius of one kilometer from this classroom. (O,M)

14. Construct a small trundle wheel out of cardboard and a fastener, as shown in Figure 5, to measure in centimeters. (M)

Figure 5



15. Using an almanac or other reference book, find the lengths in meters of various Olympic events in track and field and in swimming. (O,M)

16. Cut out three non-standard units of arbitrary length from a sheet of paper and practice measuring lengths with these new units.

17. Investigate the metric sizing for articles of clothing. (O,M)

18. Estimate in millimeters, using tools such as wrenches and sockets to check. (O,M)

19. Find the length of your average step in centimeters. (O,M)

Activities Involving Area

20. Make sets of standard square units out of cardboard, 4 square cm, 9 square cm, 16 square cm, and so on, to use in learning to estimate area in square centimeters. (M)

21. On a sheet of acetate, make a grid template in square centimeters that can be used to measure the area of rectangular regions. (M)

22. Find the areas of various irregularly-shaped regions by using a grid network marked in centimeters. (M)

23. Use a planimeter to find the area of regions. A planimeter can be secured from an engineering or drafting department.

24. Lay off one acre by having four people stand at the corners. (O)

25. Lay off one hectare by having four people stand at the corners. (O,M)

26. Use the trapezoidal method for approximating irregular areas of lawn. This method is explained in calculus texts, but uses no calculus at all (C,O,S,M)

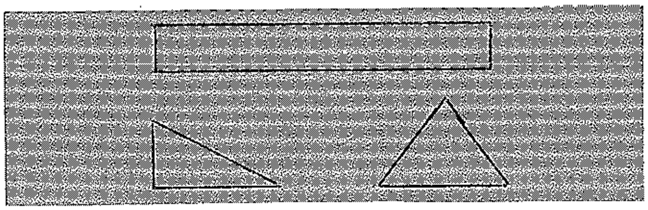
27. Use Heron's formula,

$$A = \sqrt{s(s-a)\ (s-b)\ (s-c)}$$

to find the area of a large triangular region. In the formula, s is half the perimeter and a,b,c are lengths of sides. (C,O,S,M)

28. Practice finding the approximate area of rectangular regions using non-square units such as those shown in Figure 6.

Figure 6



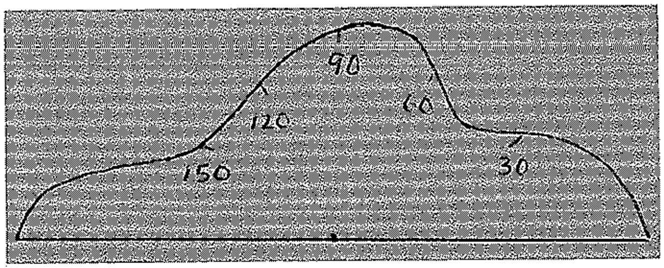
29. Draw three squares of arbitrary size, cut out the regions, then use these non-standard units to measure area.

Activities Involving Angles

30. Experiment with various ways of establishing a right angle without using a protractor.

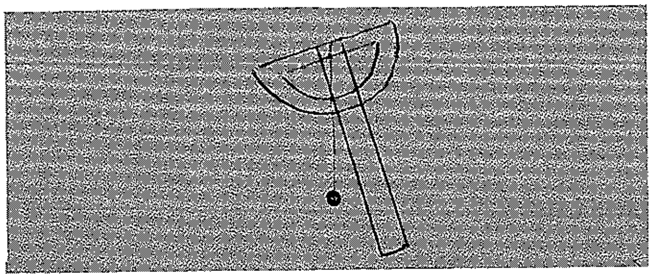
31. Construct a non-circular protractor such as the one shown in Figure 7 and practice using it.

Figure 7



32. Construct a gravity protractor such as the one shown in Figure 8 and practice using it to find angles of elevation.

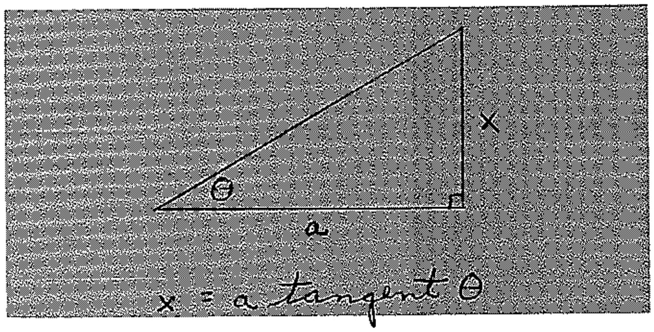
Figure 8



33. Construct a protractor for measuring in radians. (S,M)

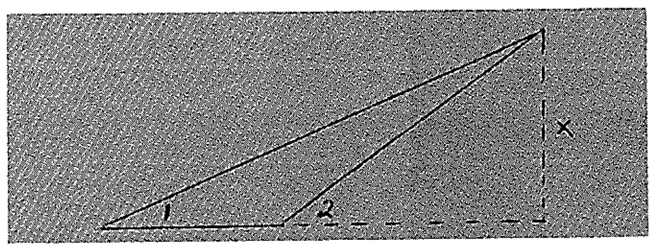
34. Use a method of indirect measurement involving the tangent of an angle to measure height, as illustrated in Figure 9. (C,O,S,M)

Figure 9



35. Use the method of indirect measurement of heights shown in Figure 10. First measure the angle of elevation, then move up a known distance

Figure 10



and measure the angle of elevation again. Construct a scaled drawing and find the unknown height from the drawing. (O,M)

Activities Involving Volume

36. Find the volume of your classroom in cubic meters. (C,M)

37. Find a reasonable value for k in the formula S = kn giving the number of desirable cubic meters of space per student in an elementary classroom. In the formula, n is the number of students. (C,S,M,)

38. Construct standard units from cardboard for the cubic centimeter and the cubic decimeter. Practice estimating volume of various objects using these standards. (M)

39. Estimate the cost per cubic meter of living space for constructing a home at today's prices. (C,O,M)

40. Use a standard liter container that is marked in milliliters. Pour some water into a tin can and try to guess how many milliliters you have poured. Check by pouring it into the standard container. (O,M)

41. If the gas stations in your area decided to sell gasoline by the liter, what would be the price per liter of regular gasoline? (C,O,M)

42. Find the volume of water in an aquarium in metric units. (C,O,M)

43. Find the volume of water in a swimming pool in metric units. (C,O,M)

Miscellaneous Activities

44. Measure the average speed of an automobile traveling down a street by timing it for a short known distance with a stop watch. (C,O)

45. Measure the average landing speed of an airplane by timing it for a short distance with a stopwatch as it approaches an airport. (C,O)

46. How hot is it outside now, in Celsius degrees? (O,M)

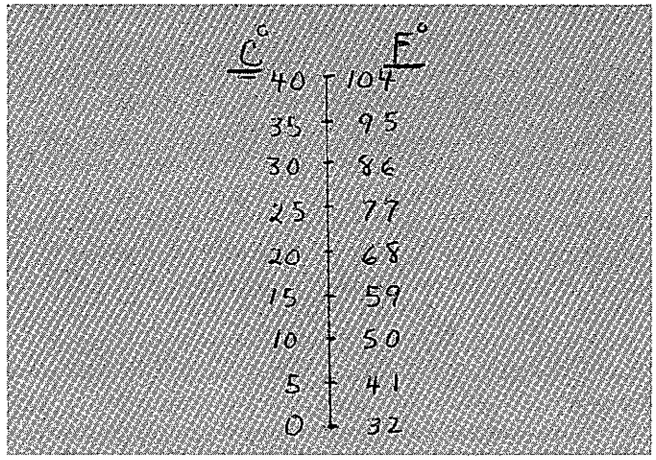
47. Construct a Celsius thermometer by modifying a Fahrenheit thermometer, using the easy conversion pattern in Figure 11. (M)

48. Take a cake recipe from a cookbook and change it entirely into metric units. (C,O,M)

49. Check the labels of various food products in a kitchen to see which are measured by mass and which by volume. If the item does not have the metric units, convert approximately. (C,O,M)

50. Play a challenge-type game in teams of two, seeing which team can get the closest estimate, following instructions such as, "Stand so that you are 20 meters apart." (O,M)

Figure 11



Probability — Magic or Mathematics?

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The concept of a finite probability model is well-defined, well-developed and well-understood. High school Algebra 2 textbooks make it very clear that a finite probability model consists of a list or set of possible outcomes of an experiment together with a set of probabilities, one of which is assigned to each of the outcomes in the list. The set of possible outcomes is called the sample space or event set while the elements of the event set are called simple events or elementary events. The probabilities assigned to the simple events must be greater than or equal to zero and less than or equal to one and the sum of the probabilities over all of the simple events must equal 1. An event is a subset of the event set. The probability of an event A is the sum of the probabilities of the simple events in A . If A and B are events, then the events, either A or B, is the event A Ω B . While the event, A and B, is the event A Ω B . It has been my experience that students can without too much difficulty grasp these ideas. The difficulty for the student often begins when he or she is asked to decide on what probabilities should be assigned to the simple events in the model. In their counting of possible outcomes students are sometimes confused by whether order matters or by whether objects are distinguishable or indistinguishable. Confusion can also arise over the notion of independence and when probabilities should and should not be multiplied. It is my opinion that a good many textbooks do not provide the student with an adequate explanation concerning why one assignment of probabilities is preferred over another. This note is an attempt to supply that explanation.

I. Coin Tossing

Two coins are tossed and the number of heads that appear is recorded. If you, the teacher, ask your students to construct the probability model for this experiment, the answer you expect to receive is model I.2. However, you can usually count of having one student ask, "If the coins are indistinguishable, why isn't model I-1 the correct answer?" The numbers 0,1,2 which constitute the even set here refer to the number of heads.

Model I.1

Event Set	0	1	2
Probability	1/3	1/3	1/3

Model I.2

Event Set	0	1	2
Probability	1/4	1/2	1/4

You can often convince the student of the error of his or her ways by asking what would be their answer if a nickel and a penny were tossed rather than two indistinguishable coins. However, this is only partly satisfactory. Model I.2 is the correct answer because the probabilities given in that model will be seen to be approximated by the long run frequencies of the three elementary results, no heads, one head, two heads if the experiment of tossing two coins is repeated on a large number of times. This can be demonstrated in the classroom by computing the frequencies that result when your students toss a pair of coins many, many times. However, a word of warning is in order. Unless you are prepared to have your students toss coins all period, you may be in for some embarrassment. During a talk which was based on this paper given at the El Paso Regional Meeting of the National Council of Teachers of Mathematics, my audience supplied me with a single head frequency half-way between one-half and one-third. Obviously in this case, 84 tosses of a pair of coins was not a large enough number of tosses.

The real world phenomenon of long run stability of frequencies is basic to the building of probability models. It must be emphasized, however, that the stability property of frequencies is not a consequence of logical deduction. In the words of J. L. Hodges and E. L. Lehmann [2], "it is quite possible to conceive of a world in which frequencies would not stabilize as the number of repetitions of the experiment becomes large. That frequencies actually do possess this property is an empirical or observational fact based on literally millions of observations. This fact is the experimental basis for the concept of probability."

II. Tossing Two Marbles Into Four Urns

Our main point can also be made by considering the experiment of tossing two marbles into four urns. This experiment might be described by any one of the following four models

Here the two digits in each of the numbers in the event sets correspond to the urns that are occupied. The elementary event 13 corresponds to the result that urn #1 and urn #3 each contain a marble while the elementary event 22 corresponds to the result that both marbles find their way into urn #2. If one of the marbles used is green and the other yellow, then students can usually be expected to answer that model II.1 gives an assignment of probabilities that is appropriate for this experiment. However, if the students initially are told that the experiment is being performed with two red marbles, then some uncertainty may find its way into the minds of your students. If the experiment is to be performed with thimble sized urns that contain room for only one marble, then either model II.3 or model II.4 is appropriate, however you might expect to get some student disagreement over which of these two models should be used. A moments reflection, though, should make it clear to your students that the last two models are equivalent. To the result that both urns #1 and #2 are occupied both models assigned probability 1/6. If the marbles are indistinguishable then we use II.3 with the simple event 12 corresponding to this outcome. If the two marbles are distinguishable then we can use II.4 with the two-element event consisting of the simple events 12 and 21 corresponding to this result. However, confusion concerning II.1 and II.2 is not quite so easy to dispel. When a double occupancy is possible, the fact that the probability of the event that urns #3 and #4 are occupied is correctly given by model II.1 as $1/16 + 1/16 = 1/8$ and not by model II.2 as 1/10, whether the marbles are distinguishable or not, can be definitely established only after an examination of the frequencies associated with a large number of performances of the experiment. However, an approach similiar to the penny-nickel argument of the previous example can usually convince the unsure, thus removing the

Model II.1

Event Set	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
Probability	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16

Model II.2

Event Set	11	12	13	14	22	23	24	33	34	44
Probability	1/10	1/10	1/10	1/10	1/10	1/10	1/10	1/10	1/10	1/10

Model II.3

Event Set	12	13	14	23	24	34
Probability	1/6	1/6	1/6	1/6	1/6	1/6

Model II.4

Event Set	12	13	14	21	23	24	31	32	34	41	42	43
Probability	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12

need for many, many repetitions of the experiment. Doubts, though, still may linger. If the marbles are indistinguishable then the event set for the experiment will have to be the event set of II.2. "What then?" your students might ask. In this situation we use the event set of II.2, but we do not assign equal probabilities to each of the simple events. Instead, we base our assignment of probabilities on model II.1, so that the simple events 11, 22, 33 and 44 are each assigned probability 1/16 while the remaining six simple events are each assigned probability $1/16 + 1/16 = 1/8$. A blindfolded experimenter will expect to find urns #3 and #4 occupied approximately 1/8 of the time whether the marbles are distinguishable or not. Both the coin tossing and marble tossing examples illustrate the general principal that in the macro world of our experiences objects behave as if they are distinguishable even though they may not appear to be so. When probability assignments are based on considerations of symmetry, this general principal should always be kept in mind.

With very much larger numbers, the previous models are used in physics. Urns become physical states and marbles become particles. Particles whose behavior can be described by models II.1, II.2, and II.3 are said to possess, respectively, Maxwell, Boltzmann statistics, Bose-Einstein statistics, and Fermi-Dirac statistics. Experimentation has revealed the surprising fact that in the micro world of particle physics there are particles that behave as if they are indistinguishable, that is, there are particles possessing Bose-Einstein statistics. Perhaps an even more surprising fact is that no known particles possess Maxwell-Boltzmann statistics. An assignment of probabilities like that in II.1 which some have argued in the past is inherent to the notion of randomness has been shown through experimentation not to be relevant to the study of the random behavior of physical particles. The fact that for appropriately defined physical states, no two electrons, neutrons or protons can occupy the same state and hence that these particles possess Fermi-Dirac statistics does not contradict our intuitive notion of randomness. However the fact that there are physical forces at work causing protons, nuclei and atoms containing an even number of elementary particles to behave as if they were indistinguishable, that is to possess Bose-Einstein statistics, does make us possibly want to look again at our concept of randomness. To what extent the physics of Bose-Einstein statistics is well understood we will leave to the physicists to debate. Our purpose in discussing the behavior of certain physical particles is only to make again the point that a "correct" assignment of probabilities in a probability model is that assignment that best reflects reality in the sense that the probabilities are well approximated by the relevant real-world long run frequencies. W. Feller [1] makes the statement "no pure reasoning could tell that photons and protons would not obey the same probability laws." With reference to Bose-Einstein and Fermi-Dirac, Feller goes on to say "the justification of either model depends on its success" and that this discussion "provides an instructive example of the impossibility of selecting or justifying probability models by *a priori* arguments."

The counting formulas used to compute the probabilities in II.1, II.3, and II.4 are, of course, a part of the material covered in most algebra 2 units on probability. If there are no n urns and r distinguishable marbles and if multiple occupancy is permtited, then there will be n^r possible distinguishable outcomes of the experiment of tossing the marbles into the urns. If no more than one marble can occupy an urn, then the appropriate counting formulas are, for II.3 $\frac{n}{r}$, and, for II.4, $n!/(n-r)!$. In each of these three cases the desiring probabilities are simply the reciprocals of the relevant counting formulas. The formula needed in II.2 is that which gives the number of distinguishable arrangements of r indistinguishable marbles in n distinguishable urns, or, equivalently, that which gives the number of distinguishable non-negative integer solutions to the equation $r_1 + r_2 + \cdots + r_n = r$. To see that this is the case think of r_i as the number of marbles in the i^{th} urn. The total number of marbles is then $r_1 + r_2 + \cdots r_n = r$. In II.2, $r = 2$ and $n = 4$. The solution $r_2 = r_3 = 1$, $r_1 = r_4 = 0$ corresponds to the occupancy of urns #2 and #3 and hence to 23 in the event set while the solution $r_1 = 2$, $r_2 = r_3 = r_4 = 0$ corresponds to the double occupancy of urn #1 and hence to 11 in the event set. Enumeration will show that there are 10 distinct non-negative integer solutions to the equation $r_1 + r_2 + r_3 + r_4 = 2$. Discovering the required counting formula can stand as a challenge to your better students. The fact that this formula is given by

$\frac{n + r - 1}{r}$ is not immediately obvious although the clever stars and bars proof of this result found on page 38 of reference 1 does provide the reader with some insight. An algebra 2 student who understands the binomial coefficient counting formula will be able to follow the Feller derivation of the $\frac{n + r - 1}{r}$ formula.

III. Independent and Sex Ratios

If A and B are two events and if the real-world outcomes corresponding to A and B are believed not to influence one another, then A and B are said to be independent and the assignment of probabilities within the model is made in such a way that $P(A \Omega B) = P(A)P(B)$. The fact that it is the long-run behavior of frequencies that leads us to associate real-world unrelatedness and the multiplication of probabilities is a point that again I feel is not always clearly made in algebra 2 textbooks. Before presenting examples designed to clarify this point, it is necessary to construct a probability model with simple events that correspond to the sex of a newborn child. Then if we label these simple events M for male and F for female, it is tempting, due to our understanding of how sex is determined, to assign equal probabilities of 1/2 to each of the simple events M and F. However, once more there are forces at work which cause frequencies to differ from the probabilities deductively obtained from symmetry considerations. Data on the sex of newborn children suggest that the assignment of probabilities in model III.1 is closer to the mark than is the assignment 1/2, 1/2.

Model III.1

Event Set	F	M
Probability	.485	.515

Model III.2

Event Set			
MM	MF	FM	FF
Probability			
$(.515)^2$	(.515)(.485)	(.485)(.515)	$(.485)^2$

Table I

2nd \ 1st	M	F	Total
M	526	498	1024
F	506	470	976
Total	1032	968	2000

Next, let us imagine that we are interested in the sex of the first two children born in your county in the new year. If we decide to ignore multiple births, then our intuitive feeling is that the sex of the firstborn can in no way affect the sex of the second born. If the data in Table I were a record of the sex of the first and second births in 2000 counties across the country for this year, then if our intuition is correct we would expect to find that the ratio 526/1032 is approximately equal to the ratio 1024/2000, that is we would expect to find that among those counties that recorded a male first birth, the proportion of male second births would be approximately the same as the frequency of male second births in all of the 2000 counties. This, in fact, is the case. The approximate equality 526/1032 = 1024/2000 can be rewritten as 526/2000 = (1032/2000)(1024/2000).
Hence, if A = {MM,MF} (the first birth is male), and B = {MM,FM} (the second birth is male), then A ∩ B = {MM} (both births are male) and the approximate equality tells us that in our model we want P(A ∩ B) = P(A)P(B). III.2 should then provide an acceptable model for the experiment of noting the sex of the first and second born children in your county in the new year.

Let us turn next to an example that exhibits a lack of independence. A child is selected at random. Its sex is noted as well as whether or not he or she is color blind. The possible outcomes for this experiment are listed in model III.3. The simple event MC corresponds to the result that the child selected is both male and color blind while the simple event FN corresponds to the result that the selected child is female and normal. MN and FC have similar interpretations.

Model III.3

Event Set			
MN	MC	FN	FC
Probability			
(.93)(.515)	(.07)(.515)	(.995)(.485)	(.005)(.485)

Table II

	C	N	Total
M	72	958	1030
F	5	965	970
Total	77	1923	2000

The assignment of probabilities in III.3 must be based on observed frequencies. Referring to Table II, the ratio 72/77 is not even close to the ratio 1030/2000. Among those who are color blind, the male frequency, 72/77 is much greater than the male frequency in the general population, 1030/2000. Color blindness and sex are related. Hence, if A = {MC,MN}, the event that the child selected is male, and if B = {MC,FC}, the event that the child selected is color blind, then A ∩ B = {MC}, the event that the child selected is a color blind male, and we do not want P(A ∩ B) = P(A)P(B). This, in fact, is the case in III.3 because (.07)(.515) does not equal (.515)[(.07)(.515) + (.005)(.485)].

Tables like Table I and Table II should, I feel, be used as teaching aids when independence is discussed in the classroom. You can form tables with numbers that are not based on actual population figures as was done in Table I and Table II or you can have your students generate actual data with, say, a coin and a die. Actual data is, of course, preferred but if you choose to use contrived figures, the numbers selected should produce frequencies that are close to frequencies that are based on actual data. The probabilities in models III.1, III.2, and III.3 are close to the frequencies based on a large number of actual observations. The important thing to keep in mind is that arrays of data like those shown in Table I and Table II provide the student with a quantitative bridge joining his intuitive notion of unrelatedness with the multiplication of probabilities within the model. The tables should help to make it clear that the independence of two events means that the frequency of the first event among those trials where the second event occurs must be approximately equal to the unrestricted frequency of the first event.

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1 Feller, William, *An Introduction to Probability Theory and Its Applications,* New York. John Wiley and Sons, Inc., 1968.

2 Hodges, J. L., Jr. and E. L. Lehmann, *Basic Concepts of Probability and Statistics.* San Francisco, Holden-Day, 1970.

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