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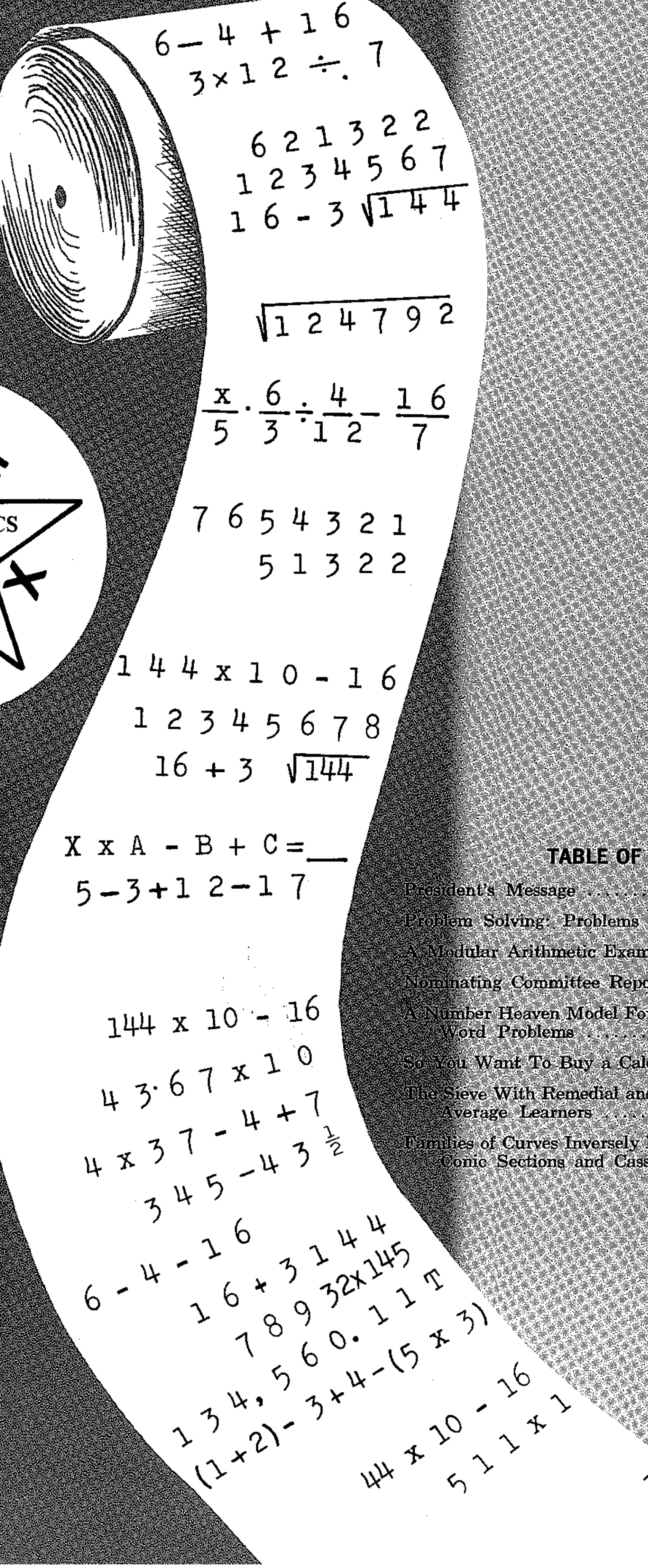

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TEXAS MATHEMATICS TEACHER is the official journal of the Texas Council of Teachers of Mathematics. The views expressed are the contributor's own and are not necessarily those of the publisher or the editor. All manuscripts and correspondence about this publication should be addressed to Mr. J. William Brown, *Texas Mathematics Teacher*, 100 So. Glasgow Drive, Dallas, Texas 75214.

TEXAS MATHEMATICS TEACHER is published quarterly by the Texas Council of Teachers of Mathematics. Payment of membership fee of $5.00 entitles members to all regular Council Publications.

President's Message

It is interesting to observe the critics of education have begun questioning reading teaching strategies rather than mathematics skills. Hopefully, this is because we have survived the trend downward in student computational skills and the pendulum is swinging upward for us and our students. For sure, with all the activity surrounding the Bicentennial, many more teachers and students are becoming involved in mathematics.

With the large, and growing, number of workshops and contests throughout the state it is becoming difficult for groups to schedule events without conflicts. By this time some groups are considering dates for next year's meetings. If you will send me those dates I will try to set up a calendar of events for program-planning coordination. This should enable all of us to participate more wisely in events throughout the state. Concerning dates — CAMT is already scheduled for October 28-30.

As you are aware the last issue of the journal was late. This was in no way the responsibility of the editor. I feel we are very fortunate to have J. William Brown doing the editing for us with much dedication. Perhaps you might write him to express your appreciation if you share this opinion.

For some of you this will be the last journal you receive. Membership reminders have been sent twice and we can no longer carry unrenewed memberships. At the mailing of this journal, the membership list will be revised and all names deleted where membership is not current. Please, if you have received a reminder, take time to return it now. There are some 200 members in this catagory.

Included in this issue is a list of nominees for offices to be filled for next year. Each office has at lease two nominees. Please mark your ballot and return it immediately. The nominating committee has done a great amount of work to submit these candidates to you. It is important that you share in the election process. After all, this is your organization.

It is extremely difficult for communication lines to exist across the State. Should you have suggestions for improving our organization, please let me know. Only as all of us are involved does our existence exhibit meaningfulness.

In a recent issue of a newsletter from another state I noted one interesting area was contributions from teachers describing what they were doing and how it was succeeding. If you have something to share along the line of "what we are doing and how it's working," please send it to us.

BILL ASHWORTH

Problem Solving: Problems and Story Problems

by Randy F. Elmore and William D. McKillip
The University of Georgia

Skill in problem solving is an important objective of all phases of elementary education. Defined as the ability to apply previously learned principles to analogous new situations (Gagne, 1970) or to have insight in finding solutions to new problems (Ausubel, 1968) problem solving skills are essential for students to inquire into the various subject areas of the elementary school. In the paragraphs which follow we describe some practical techniques for teaching problem solving in school subjects and in interpersonal relations.

Too often, children and teachers alike are unhappy about their progress in problem solving. Instead of enjoying the activity of solving interesting problems, they feel only failure and frustration. We want to present here some practical suggestions for beginning an *"up-cycle"* from success to enjoyment to interest and avoiding a *"down-cycle."* Many of these suggestions we have learned from watching teachers at work. Some come from other sources such as research and psychology. In any case, all these suggestions have been organized into a program to help children improve their problem solving skills. This program has been tried many times and almost always produces the desired results.

INTEREST
↑
ENJOYMENT
↑
SUCCESS
Up-Cycle

Down-Cycle
FAILURE
↓
FRUSTRATION
↓
AVOIDANCE

Start a program for problem solving. You can help your students improve their problem solving skills if you will plan a long range program of activities in the different subject areas. An important goal such as problem solving skill cannot be attained through occasional or unplanned instruction. Here are some specific suggestions for implementing your program. Many of the suggestions for mathematics are taken from "Teaching for Problem Solving Skill," (McKillip, et al, 1975).

***Present one or two problems each day.** Lack of continuous practice in solving problems is one cause of children's difficulties in this area. If you

*This is the most important single suggestion for success in developing problem solving abilities.

do one or two a day your students will become familiar with them and more at ease in attempting to solve them. The following examples in mathematics, social studies, and science illustrate some interesting ways to present and solve problems.

Presenting Problems in Mathematics. Presenting a problem in mathematics means more than just giving an assignment. When you "present" a problem you should first give each student an opportunity to work on it. Then you should . . .

a. make sure all students understand the problem by reading it aloud, asking students to read the problem, and asking the students questions about the problem. With younger children it may be helpful to act out the problem in class.

b. relate the *actions* in the problem to the operations to be used. It is this connection between action of things and the abstraction of mathematical sentences which is at the heart of solving story problems and, of course, is important in all mathematics learning. Do *not* use special words, for example, do not say "This is a subtraction problem because you see the word 'left'."

c. have one or more students show their solution and explain how they got it. Discuss why the equations they used are appropriate for the actions in the problem.

d. check the work in a variety of ways. Ask if the computation is correct, if the answer is reasonable or realistic, if it makes sense. Ask what the answer means. If 7 is the answer, 7 what? Show how the answer fits the problem.

For most problems this sequence of activities takes only a few minutes. These few minutes will pay off in increased skill and enjoyment on a topic which is traditionally a difficult and frustrating one for children.

Solving problems in social studies. Since social studies involves a number of different disciplines, the nature of the problem solving activities may vary. However, the basic steps of problem presentation, hypothesis making, and verification listed above for mathematics can also be used for solving social studies problems.

If you are studying an area of content such as history, geography, or economics, a very dramatic way to present and solve problems is through "role playing." The setting and solving of such a problem may occupy a class for several days as they collect and organize the information needed.

In the instance of pioneers traveling westward across the rocky mountains, for example, there was a constant problem of providing food. Either they had to take it with them or they had to obtain it along the way. Let's use this dilemma to illustrate some steps that can be used to solve a subject matter related problem.

a. First, you must work with the children to create the background for playing. If the pioneers are in a wagon train, how many wagons are there? How many people? Where are they traveling and how long will the trip take? How much food can be hauled in the wagons and how much will have to be obtained along the way? The roles of the children will have to be defined and then the problem can be acted out.

b. To some extent the children will begin to make hypotheses as they build the background for the play and guess about the amount and kind of food that will be needed. Then once they begin role playing they will speculate further about ways to solve the food problem.

c. To find out if the role playing produces realistic solutions, the children will have to do some research. This will involve reading, consulting with experts, and discussing the accuracy of the plans.

While some social studies problems are related to subject matter others involve interpersonal relationships. "Name calling" among pupils is an example of this type of problem. To try to solve such problems . . .

a. have the class sit in a circle and establish a rule that each person is to avoid making judgments about what is said. Then, lead the group to pinpoint the problem of "name calling."

b. After the problem is clearly defined, guide the children to explore possible causes and solutions. Each student should be encouraged to participate and contribute his feelings and ideas.

c. Verifying the solution to an interpersonal problem may be more difficult than checking historical facts. However, if there is a concensus of the group on alternative solutions and if the group shows commitment to the solutions, the problem will probably be solved. The ultimate test will be to wait and see if the children stop "calling names."

Presenting problems in science. As in mathematics and social studies, students should spend time on a daily basis solving science problems. Only in this way will students become comfortable and more interested in solving science problems. The steps that should be followed include defining the problem, gathering data, making hypotheses, conducting experiments, and drawing conclusions.

Whether the problem solving activity in science is teacher directed or discovery oriented is a matter of judgment. You will have to decide whether to use students' questions or your own as points of departure. There is some evidence that students will be more highly motivated if they are trying to answer their own questions.

To work through the problem solving process in science let's assume that some students want to know, "what causes rain?"

a. After defining the problem, the students should gather as much information about rain as possible. They should observe rain and make notes about everything they see. These notations can include comments about cloud color and formation, temperature, wind, etc. — everything that can be noticed about the rain.

b. Next a list can be made of things they think make rain. You may need to guide their thinking toward the concepts of *evaporation* and *condensation.*

c. Then ask how they might experiment to discover if their guesses are correct. Again, you might have to guide their planning. The age and experience of the children will determine the amount of guidance.

d. Assuming that you conduct experiments such as placing water in the sun to illustrate evaporation and placing ice inside a jar to demonstrate condensation, the next step would be to lead the students to draw conclusions about the way rain occurs.

In each area of study, the more you are able to get students to participate — to become actively involved in discussing ideas, manipulating materials, and reading for discovery the better your problem solving activities will be. Also the more you are able to motivate students to ask questions, conduct experiments, and draw conclusions without your direction the more you and your students will move toward an *"up-cycle"* of success, enjoyment, and interest.

Start with very easy problems. If your students are having difficulty solving problems you may have to use problems usually found at lower grade levels. You may also need to change the problem so that the context (social setting) is familiar, or you may need to use shorter sentences and simpler words. In mathematics, it frequently helps a lot to take the problem and make the numbers *very small.* When students have solved the "small number" version they can often go back and solve the "big number" version. It is essential to start with problems the students can solve successfully. Only in that way can the *"up-cycle"* begin. When students are working successfully and are enjoying the activities you can increase the difficulty of the problems.

Be sure your students understand or acquire the knowledge that is necessary to solve the problem before you attempt a solution. In mathematics use problems which require operations the children can perform. This puts the emphasis on understanding the problem and planning how to solve it. A child who cannot yet do long division will naturally be frustrated by a story problem which requires long division in its solution. And, while there are fewer prerequisite understandings for solving problems in social studies and science than in mathematics, you must make sure that students understand concepts and principles that are necessary to find appropriate solutions. Students must understand the concepts of *condensation* and *evaporation* before they can begin to understand how rain occurs.

Use a variety of problems. In each subject area, avoid presenting only one type of problem or presenting only the problems listed in the book. When working in mathematics, use problems which require many different operations. Part of the story problem solving process must be deciding what operations or operations to use. If all your problems require division (because you are studying division) the children will just divide, not read and think. Another way to introduce variety in mathematics is to use different situations which lead to the same operation. In the three problems presented here you will note that they all lead to the same equation: $12 - 7 = \square$. The variety is in the action taking place: take away, partition, and comparison.

1. John has 12 marbles. He lost 7. How many does he have now.
2. John has 12 marbles, some blue and some red. Seven are red. How many are blue?
3. John has 12 marbles and Sam 7. How many more marbles does John have?

Variety is also essential for successful problem solving in Social Studies and Science. Presenting problems in different ways and emphasizing different processes in finding a solution are ways of achieving this. Problems can be dealt with in a number of ways ranging from student initiated inquiry and projects to teacher guided discussions, films, role playing, and simulation. Different processes such as comparing, and contrasting, defining, classifying, analyzing, evaluating, and generalizing can also be emphasized to create variety.

It is important that students in the elementary school learn to solve problems in creative ways. To accomplish this goal teachers must help students be successful, enjoy, and become interested in solving problems. We have made several suggestions to aid in teaching problem solving. Children must be involved in a long range program of problem solving activity which occurs on a daily basis. An effective problem solving program must begin with very easy problems — ones which assure success and enjoyment. Finally, a successful program, will provide for a large amount of variety in problem solving activities. If these suggestions are followed, teachers and children will move into an *"up-cycle"* of success, enjoyment, and interest in solving problems in the different subject areas in the elementary school.

Here are the key suggestions for improving children's problem solving skills:

1. Start a problem solving program
2. Present one or two problems each day
3. Start with easy problems
4. Use a variety of problems.

Bibliography

(1) Ausubel, David P. *Educational Psychology: A cognitive view.* New York: Holt-Rinehart & Winston, Inc., 1968.

(2) Gagné, Robert M. *The conditions of learning:* New York: Holt-Rinehart & Winston, Inc., 1970.

(3) McKillip, William D., Wilson, James W., Davis, Edward J., & Cooney, Thomas J. *Teaching for problem solving skill,* unpublished, mimeographed, Athens, Georgia, 1975.

A MODULAR ARITHMETIC EXAMPLE

Joe Dan Austin, *Emory University*

Many textbook applications of modular arithmetic are found in clock arithmetic or in the theory of bases. These need not be the only applications as the following problem shows.

The basic problem is to find all positive integers whose square has the same tens and units digit. For example $12^2 = 144$ and $10^2 = 100$ are two such numbers, but are there more? Let us write the number as

(1) $n = 10\alpha + j$

where j is the units digit and α is the other digits. For example if n is 312 then $n = 10 \cdot 31 + 2$. For n^2 to have the same tens and units digit — say digit r — then $n^2 = 100\beta + 10r + r$. For example $12^2 = 100 \cdot 1 + 10 \cdot 4 + 4$. Equating this to the square of n in (1) gives the basic equation

(2) $n^2 = 10^2\alpha^2 + 20\alpha j + j^2 = 100\beta + 10r + r$.

In mod 10 both $10^2\alpha^2 + 20\alpha j$ and $100\beta + 10r$ are 0, so (2) becomes

(3) $j^2 = r \pmod{10}$.

Recall that both j and r must be digits. If we let j take all the values 0,1 . . . , and 9 and note the values of r in (3), the only possible values for r lie in the set [0, 1, 4, 5, 6, 9]. (This proves that squares must end in one of these digits.) Since both the tens and units digit must be the same for the square we will be able to further reduce the possible values for r.

If n is even, $n^2 = 0 \pmod 4$ and if n is odd, $n^2 = 1 \pmod 4$. Because $10^2\alpha^2 + 20\alpha j$ and 100β are 0 in mod 4, equation (2) in mod 4 is

(4) $n^2 = j^2 = 10r + r \pmod 4$.

Since n^2 must be either 0 or 1 in mod 4, equation (4) gives that $10r + r$ must also be 0 or 1 in mod 4. For the set of possible values of r we have $00 = 0 \pmod 4$, $11 = 3 \pmod 4$, $44 = 0 \pmod 4$, $55 = 3 \pmod 4$, $66 = 2 \pmod 4$, and $99 = 3 \pmod 4$. Thus the only possible choices for r are the digits 0 and 4. We have now shown that if the square of a number has the same tens and units digit then the digit must be either 0 or 4.

We first consider the numbers whose square ends in two (or more) zeros. It is easy to show that the only such numbers are those with 0 as units digit. In (3) for $r = 0$ the only digit solution for j is $j = 0$. Conversely, in (2) if j is 0 then $n^2 = 10^2\alpha^2$ which ends in two zeros.

We next consider the more interesting case when the square of the number has the tens and units digit both 4. Lettering $r = 4$ in (3) we see that there are two solutions for which j is a digit, namely $j = 2$ and $j = 8$. Both solutions will lead us to numbers whose square ends in 44. First we consider the case when $j = 2$. Equation (2) becomes

(5) $n^2 = 10^2\alpha^2 + 40\alpha + 4 = 100\beta + 44$.

After subtracting 4 from the two sides of the right equality and dividing by 10 we have

(6) $10\alpha^2 + 4\alpha = 10\beta + 4$.

In mod 10 equation (6) is simply $4\alpha = 4 \pmod{10}$. This has two solutions mod 10 for α, namely $\alpha = 1 \pmod{10}$ and $\alpha = 6 \pmod{10}$. This means that we have found two types of numbers whose square ends in 44, namely numbers that end in 12 and numbers that end in 62. Some examples you can check are 12, 812, 62 and 1962.

The final case to consider is when the number ends in 8 ($j = 8$) and the square ends in 44 ($r = 4$). For this case equation (2) becomes

(7) $n^2 = 10^2\alpha^2 + 160\alpha + 64 = 100\beta + 44$.

Subtracting 4 and dividing by 10 yields

(8) $10\alpha^2 + 16\alpha + 6 = 10\beta + 4$.

In mod 10 this becomes simply $6\alpha + 6 = 4 \pmod{10}$ since both $10\alpha^2 + 10\alpha$ and 10β are 0 in mod 10. If we note that $-2 = 8 \pmod{10}$ the last equation is

(9) $6\alpha = 8 \pmod{10}$

There are two solutions to (9) in mod 10, namely $\alpha = 3 \pmod{10}$ and $\alpha = 8 \pmod{10}$. We now have two more types of numbers whose square ends in 44. These are numbers that end in 38 and numbers that end in 88. Some examples to try are 38, 138, 88, and 588.

Summarizing the results obtained, the only numbers whose square has the same tens and units digits are those numbers that end in (i) 0, (ii) 12, (iii) 62, (iv) 38, or (v) 88.

NOTICE:

Nominating Committee Report:

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A Number Heaven Model For Solving Word Problems

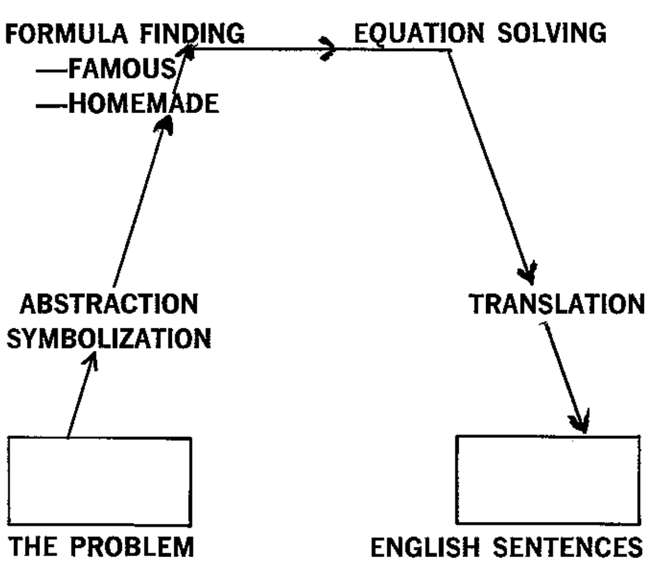
by Kenneth A. Retzer
Illinois State University

Many students regard word problems with fear and dread. Some teachers who do not share those same emotions at least feel discouraged with occasional pangs of despair. Their diagnoses, however, seem too superficial. While they frequently say that students "can't read" or "can't think," these diagnosis make it virtually impossible for them to provide appropriate remediation. The purpose of this article is to share a model for solving typical word problems which appear in mathematics texts. It is a model which I have developed and found to be helpful in mathematics classes ranging from junior high to collegiate levels.

Just as the teacher diagnoses "can't read" or "can't think" are too superficial to be followed by appropriate remediation, some students have tended to over-generalize their difficulties and express "I can't work word problems." They are unable to discern steps wihch would be helpful in developing this ability. By sharing this model with my students I have been able to diagnose and provide remediation activities, and my students have been better able to pinpoint a source of difficulty they experience.

Analysis of the skills which are used to solve mathematical word problems results in a model which might be illustrated by Figure 1. Thus, we can characterize solution of a mathematical word problem with the steps: (1) abstraction and symbolization, (2) formula finding, (3) equation solving and (4) translation. Each of these steps will be discussed, in turn, and illustrated with a sample word problem.

FIGURE 1



ABSTRACTION AND SYMBOLIZATION

While abstraction and symbolization are distinct intellectual processes, they are listed together because there seems to be no unique way students sequence the abstracting and symbolizing necessary to solve problems. For illustration, consider the word problem:

> A rectangular garden's length is two meters longer than its width. Its area is 48 square meters. Find the length and width of the garden.

We want the student to abstract relevant quantities from these words regardless of whether he symbolizes them with numerals, variables, or mathematical expressions using both. In a very true sense reading is involved in the abstraction skill. If a student cannot abstract because he cannot comprehend the sense of the physical situation or the question asked and if he is educable, a mathematics teacher might properly have reason for discouragement as well as reason to ask for the aid of a special education teacher or a reading specialist. If, on the other hand, abstraction is difficult for the student because he does not understand one or more of the terms: "length," "width," "area," "longer than," then remediation is within the capabilities of the mathematics teacher.

One difficulty with helping a student develop the discriminating ability needed to abstract the proper things from a word problem is that many textbooks provide exactly the necessary and sufficient information needed to solve each problem. Teachers who feel strongly about developing this abstracting ability can develop paragraphs which contain inadequate or extraneous information so that students can gain experience in determining necessary and sufficient information.

In our sample problem we want the student to be able to recognize that the numerals "2," "48," and the words "length," "longer than," "width," and "area" are sufficiently important to be abstracted and symbolized in expressions designating quantity or else relationship because they are necessary to solving the problem by conventional methods.

Basic to the process of solving word problems is the symbolization technique of letting a variable represent a quantity, such as "w represents the width," and writing other quantities as numerical expressions involving that variable, such as "w + 2 represents the length." Should symbolization be a source of difficulty for the student, teachers can provide practice in writing expressions involving variables which express relationships among quantities. Several textbooks contain excellent exercises which help students practice this skill. If not, a teacher can easily compose worksheets on which students symbolize such expressions as "twice a number," "7 less than a number," and "four more than ⅓ of a number."

Inherent in these specific remediation suggestions is the general suggestion that students should not be permitted to fail to solve any particular word problem without the teacher diagnosing the missing component skills and teaching these skills. In this manner the student can pinpoint and correct his difficulties before he forms negative attitudes toward word problems in general.

FORMULA FINDING

While it is trivial to note that prior to solving an equation a student must be able to write an equation which expresses the relationship among the quantities abstracted and symbolized, it is important to recognize that "setting up an equation" is preceded by a formula-finding step. This step results in an entire system of equations or inequations in more sophisticated word problems.

Formula finding can be broken down into the two cases: (1) famous formula finding, as in our example, the formula "$A = L \times W$," and (2) homemade formula finding. If diagnosis reveals that a student cannot write an equation because he cannot select the appropriate famous formula, a list of formulas might be provided to those students; this would essentially transform the divergent thinking task of an open-ended formula search to the convergent thinking task of locating an appropriate formula from among the ones listed. If the student did not learn basic mensuration formulas in the elementary grades, appropriate learning activities can be designed or taken from existing individualized modules on measurement. Common formulas for measurement of perimeters, areas, and volumes of geometric figures are frequently used along with $I = PRT$, $D = RT$, and some of the more common scientific relationships including those of direct and indireet variation.

The notion of equivalent formulas is an important preliminary consideration for some students. It is all too easy for a teacher to recognize "$A = L \times W$" as the appropriate formula in our example and expect a student to write $w(w + 2) = 48$ as the equation without realizing that this equation, which is most mathematically beautiful, is an instance of the equivalent formula "$W \times L = A$" and not the original famous formula.

Homemade formula finding is, perhaps, most closely related to the superficial "can't think" diagnosis because there are no well-known formulas to express the multiplicity of relationships among mixtures, monetary units, work units, variations, proportions and ages which appear in word problems. If this word problem-solving step were diagnosed as a difficulty of the student, it seems that nothing short of a one-on-one interactive situation could hope to increase the student's ability to compose formulas which symbolize relationships among things.

EQUATION SOLVING

Teachers report the tendency for some students to insist on using two variables in symbolizing a problem such as our example, apparently so as not to have to go to the work of writing length as an expression using w. A suggestion is made that we permit the student to use a second variable, say "Z," rather than inhibit him initially. Allow him to go through the formula finding stage which would result in a system of two equations in two variables and allow him to see for himself that he does not have the equation solving expertise to solve the system of equations which results from his choice of symbolization.

In our example we suggested that the equation to be solved would be "$w(w + 2) = 48$." This equation contrasts with what might seem to be an alternative, namely "w meters $\times$ (w + 2) meters = 48 meters." In fact, my colleagues christened my model as a "Number Heaven Model" zecause of my insistence that labels, units of measures and other such relics of the psysical world are not among the things abstracted, symbolized, and, hence, used in solving the resultant equation. The abstraction and symbolization transform the problem information from the level of the physical world to the "number heaven" realm of real numbers and, thus, the techniques used in solving the equation are the same as solving any other equation with respect to the reals.

While recognizing that physics and other sciences use dimensional analysis to (1) help find correct labels to affix to answers of scientific word problems, (2) to prevent mathematical operations on numbers from two different measuring systems such as English and Metric, and (3) to prevent operations on various magnitudes of units within one system, such as millimeters and meters, equation solving is most fruitfully regarded as being in the realm of real numbers. It makes no more sense to indicate multiplication of w meters times w + 2 meters to get w (w + 2) square meters than it does to consider two rows of eggs in a rectangular carton with six eggs in each row and claim there are 12 square eggs in that container! Consequently, when one solves the equation of our example with respect to real numbers, the resultant solution set is $\{-8, 6\}$.

If diagnosis using this model reveals that equation solving is a point of difficulty, then it is obvious that more experience with equation solving techniques is indicated. In fact, it appears from the sequence in many freshman algebra books that word problems are introduced to provide equation solving practice. If this is an intention of an author, this number heaven model may cause him to consider that equation solving isn't the only prerequisite to solving word problems. And he may want to provide for other prerequisite skills such as abstracting, symbolizing, and formula finding before students are asked to use their equation solving expertise in working word problems.

TRANSLATION

Translating information from a solution set to English sentences which answer the word problem has an important advantage. Before this problem solving model was developed and shared, some students would write "—8, 6" as a final answer to our sample problem. When I directed them to label their answers, other students would write "—8 feet, 6 feet" as a final answer. Insisting that answers to word problems be labeled, found me in a continuing, hopeless battle, similar to that of teachers who insist that rational numbers always be expressed in lowest terms and that improper fractions always be expressed as mixed numerals. However, pointing to the necessity of the translation step has helped with the labeling of answers.

Discussions of abstracting numbers from descriptions of physical situations to use in equation solving in "Number Heaven" provide a basis for students to regard the resulting solution set as nothing more than a set of real numbers usable in translation to sentences about the physical world again. Noting that the equation was expressed in w, which symbolized width, one can see that while —8 is a mathematically correct solution to a quadratic equation in real numbers, only 6 makes sense with respect to garden width. Furthermore, is it argued that since the problem was posed in the English language the "polite" response would be to express an answer in the English language. Many times an analogy to this sense of propriety is drawn with the Goofus and Gallant characters of the magazine *Highlights for Children* with which many present day students are familiar. That sense of propriety would encourage students, working our example, to write something like "The width is 6 meters." which automatically takes care of the labeling problem. A reexamination of information sought in the problem will indicate that a complete response should also include a sentence something like "The length is 8 meters." Insisting on English sentences, however, is not a foolproof solution to the labeling dilemma, for a few students still write a sentence like "The answers are 6 and 8." and contend it technically meets the requirements of the number heaven model.

This "Number Heaven Model for Solving Word Problems" can be shared with your students. It is intended to provide both students and teacher with a sufficient diagnostic structure to pinpoint the type of remedial help needed when attempts to solve a word problem fail. Hopefully, we will be able to replace statements like "I can't solve word problems" with "I can't solve *this* word problem because I cannot symbolize . . . ," or "I'm stuck with *this* problem because I cannot find an applicable formula." The implications of this kind of statement rather than a general statement of defeat certainly seems to have implications for the attitudes of our students toward word problems as well as for their ultimate ability to solve many of them.

So You Want To Buy a Calculator

by Daniel T. Dolan and James Williamson
Columbus High School
Columbus, Montana

Math educators throughout the country are faced with a new crisis in the classroom, the hand calculator. Prices have decreased dramatically in the face of widespread inflation and continued lower cost coupled with increased sophistication is predicted. It is now possible to find a simple calculator at the same or lower cost than that of a math textbook.

Given these facts, and the more than thirty million devices now in use, teachers have begun and must continue carefully designed experimental classwork with the hand calculator.

If teachers intend to begin this work with students the choice of machinery becomes a fundamental problem. Selection today is varied and prices fluctuate almost daily. Thus far, teacher input has not been significant with respect to the machine features most desirable in the classroom, so we must take what the manufacturers produce.

Educators are faced with such choices as fixed or floating decimal, arithmetic and algebraic logic, memory, constant factor, and many others.

What kind of machine should be chosen?

What capabilities should it have?

How does one analyze machine functions so that a proper selection is made?

During the summer of 1975, eleven junior high teachers from Montana were involved in writing and developing materials for a Title III Project entitled "Math Lab Curriculum for Junior High" at Columbus, Montana. A segment of this program includes materials and activities utilizing the hand calculator.

As the program was to purchase 100 calculators for pilot schools, several firms made calculators available to the team for analysis and testing.

Wallace Judd, who has published extensively concerning the hand calculator, worked as a consultant with the writers for three days. He was extremely helpful in directing careful analysis of each calculator. As members of the team worked with a variety of available machines, it was obvious that features varied widely, even where keys were identical.

After careful analysis, the teachers involved concurred that anyone considering purchase of calculators for schools should spend considerable time in studying each machine and its functions. The following features were found to be most desirable for use with junior high students.

1. **Algebraic logic.** Most calculators today are begin produced with this feature as opposed to arithmetic or adding machine mode. Algebraic logic allows the problem to be entered as it is written. For example, $5 \times 2 + 6 = 16$ can be solved by pressing the keys in exactly that order, similiarly $14 - 3 = 11$ or $24 \div 6 \times 2 \div 5 = 13$. Laws of operation must be considered in problems such as $6 + 2 \times 5 =$. It must be entered as 5 ⊗ 2 ⊕ 6 ⊜.

A machine with arithmetic logic is immediately distinguished by the (±) (∓) keys. An addition problem such as $5 + 3 + 7 =$ must be entered as 5 (±) 3 (±) 17 (±). A subtraction problem such as $17 - 5 =$ is entered as 17 (±) 5 (∓), however, it can be entered as 5 (∓) 17 (±). The latter may possibly lead to the suggestion that subtraction is commutative.

The algebraic logic is most consistent with the natural way that a student thinks and does mathematics. Thus, it would seem more appropriate for increasing understanding, rather than introducing a new mode of operation along with the calculating tool.

2. **Full floating decimal.** Most of the newer calculators are being constructed with this capability. One can check this feature quickly by entering $17 \div 3 =$. If the answer is displayed as 5.6666666 on an eight digit display, then the machine has a full floating decimal. If, however, 5.6666667 is displayed, the machine is rounding off in the last place.

Some calculators, those that contain a selector switch with F, O, 2, 4, allow both floating and fixed output. One should check these outputs with a selector at each position to find out if the calculator is rounding off the answers.

The capability of selecting the output mode can be very desirable in the junior high classroom. In many situations, we may wish to use an integer output along with a decimal input. For business or monetary problems, the round off at two decimal places can also be useful. Analysis of repeating patterns in conversions of fractions to decimals necessitate the full floating decimal.

A caution is necessary here concerning the calculators that claim twelve or sixteen digit display with the use of special key (↑). On one such machine, any problem such as $17 \div 3 =$ results in a terminating decimal. The answer displayed is 5.33333 (lower register), 330000 (upper register). The final six digits are displayed by pressing the (↑) key. Thus, *all* rational numbers result in terminating decimal in the eighth place.

3. **A separate clear entry and clear key.** We found that some machines have only a (C) key and if a mistake is made with a single digit entry, the entire problem must be cleared and one must start over. The clear entry key (CE) allows one to clear only that which is on the display, and not destroy other data which has been entered into the machine. This can be of great value to students that have not yet mastered the keyboard.

Some calculators with eight digit display will indicate an error if the number entered, or the result of the operation, exceeds eight digits. Others will display the answer with the decimal point in the correct place, and indicate that the number must be multiplied by 10^8 or some other power of ten. The latter machine is certainly more appropriate. In both cases, most machines we investigated are then frozen, that is, no further calculation can be done until the machine is cleared with the (C) key, thus, clearing the entire problem.

On one machine, however, we could press the (CE) key, the overflow indicator would vanish, and then we could continue the operations. It must be remembered, however, that the displayed number is still a multiple of 10^8. One might think that this feature has little or no value, but we investigated several instances where it became a definite advantage.

4. **Constant factors.** Be careful of this one. Some older models have a (K) key for this purpose. Thus, the student must remember to set it. He may also find that the constant is stored in the memory and this will affect his use of both the memory and a constant. Again, one must be careful as we found machines where the first factor became a constant for one operation and the second for another.

This is easily checked by doing the following:

Press	6	(×)	5	(=)	30	5, the second factor, is constant for multiplication
Press			7	(=)	35	
Press			1	(=)	5	
Press	15	(÷)	5	(=)	3	5 is constant for division
Press			20	(=)	4	
Press			30	(=)	6	

Any machine used in the classroom should be consistent using the same factor as a constant for all operations. It seemed most appropriate to us that the second factor should be the constant. This alleviates possible confustion with respect to the commutative property for subtraction and division.

5. **Change of sign key** (+/−) . This is most desirable if one wishes to use calculators with integers. Care should be taken to note the location of the minus sign on the display. Some models place it to the right, others have a small light indicator. Our concensus was that it is most desirable for students to have it placed to the left of the display just as it would be written on paper.

We found one calculator with upper and lower registers that indicated the minus sign to the left, sometimes. However, the result of multiplying a large number (five digits or more) by a negative number was displayed as a positive number. The minus sign was displayed only if the (↑) key was depressed, and then it was indicated on the upper register to the right. A student using this machine might easily get the idea that the product of a positive integer and a negative integer is positive, especially, if you use very large positive or very small negative numbers.

6. **A memory.** While this feature is not an absolute necessity for junior high or elementary students it can certainly save a great deal of time in more complicated problems. Some machines have a simple store (STO) and recall (RM) key, whereas others have an accumulating memory (M+) (M−) . The latter will add or subtract the displayed number to the contents of the memory with the (M+) key or (M−) key.

Once again, be careful in analyzing calculators with the memory feature. We found one machine with a functioning memory, (M+) and (M−) , that added the answer obtained when using the equal key automatically to the memory. If the (M=+) key was then used the result was added a second time. Thus, if one pressed 3 × 2 = 6, six was entered into memory. If (M=+) key was then pressed, twelve would be the result in memory.

This particular method of memory usage could be most confusing to young students. They would certainly wish to see the results of each step in a problem and then enter it into the memory. It can also cause trouble if one forgets, and presses the equal key before the (M+) key.

If a memory machine is to be purchased, be sure that it contains a separate clear memory (CM) key. Some calculators we analyzed had one key to clear the machine and the memory (C/CM). One press cleared the machine and a second cleared the memory. We found this to be a definite disadvantage. If a problem involved several calculations which were to be placed in memory, one had to be sure that the memory was clear before starting. At various times, we found ourselves clearing the problem and the memory because of lack of foresight. Students would certainly have this same problem.

Another feature important on a memory machine is the memory indicator. Students may forget from one problem to the next that data has been stored. Thus, a light or display indicator of some kind is most beneficial.

A final consideration in the purchase of hand calculators is the power source. Two types are now available, those with a built in recharging unit and those without. The latter utilizes two-four penlight batteries that may or may not be rechargeable.

Most all hand calculators can be purchased with A-C adaptors. A decision to use adapters in the classroom should take the following into consideration:

1. Availability of outlets in the room.
2. Desirability of numerous loose wires for students to trip over.
3. A lack of flexibility in the use of the machines if they are to be constantly plugged in.

At this time, teachers involved in the project are testing various types of power supply. We do not feel that specific recommendations can be made yet as to which is most desirable. By the end of this school year, sufficient data should be available for evaluation.

Each of the teachers involved in this writing project had previous experience with calculators. Some had machines in their classrooms, while others had investigated various models for purchase in their district.

After our sessions with Mr. Judd, and further analysis following his departure we concurred that we had learned a great deal more about investigating machine capabilities.

Since then, each of us who looks at a new model calculator, will carefully scrutinize each feature claimed by the manufacturer. We will also make certain that those capabilities are consistent with the mathematics being taught and the level of understanding of the students who will use the machines.

Teachers and administrators desiring to purchase calculators will serve themselves and their students well to take ample time in the same type of analysis.

The Sieve With Remedial and Below Average Learners

By Harold N. Flinsch and Anthony C. Maffei
Dreher H.S., Columbia, S.C.

It is not uncommon to find in textbooks designated for average and above average high school mathematics students digressions on the nature of prime numbers. This, unfortunately, is not always the case in textbooks designed only for remedial and below average students. Using a variation of the sieve of Eratosthenes, teaching primes to remedial and below average high school learners can be the source of a constructive diversionary activity from daily mathematics classes for the following reasons:

1. Obtaining primes strengthens skills in basic computational operations.
2. The idea of divisibility of integers is introduced.
3. Finding primes provides for a different and unique challenge in learning mathematics.

Before the teacher begins with worksheets similar to the tables, it would be beneficial for him to introduce as motivation a discussion on the nature of a prime number and a little history surrounding the past and present efforts at obtaining primes. Names and facts such as Eratosthenes, or the largest verified prime, so far containing 6,002 digits, might attach some significance to the student's work.

As preparation, the teacher might write on the board the integers from 1 to 10. He asks if any of them, excluding 1, can be individually divided only by the number itself and 1. The peculiarity of these integers should stimulate interest in finding other such integers beyond 10. A formal definition of a prime number can now be written down, as well as a working definition of the term multiple.

Each worksheet, which can be made up by the student or distributed by the teacher, should test primes between any given integers, for example, from 1 to 100, then from 101 to 200, etc. The first worksheet rests on the student's awareness that 2, 3, and 5 are the first three primes. From these three, most all primes can be found. On the left side of the worksheet the student crosses out all multiples of the first three primes according to the three rules at the top of the worksheet. The teacher now points out that 49 was not crossed out since it is not a multiple of 2, 3, and 5. However, it is a multiple of the prime 7 and as a result the student will have to cross out the multiples of the remaining uncrossed numbers, starting with 7. Similar additional steps will secure all primes from 1 to 100. In order to do the next worksheet (from 101 to 200), the preceding worksheet must be used as a reference for carrying over the last multiples of each prime. Work can be minimized in the latter worksheets when the student counts and crosses out every 7th, 11th, 13th, 17th, and 19th multiple instead of listing them first. Students should then be encouraged to discover their own short-cuts in crossing out multiples of primes. This will strengthen their understanding of prime numbers. Some will begin to see why the multiples of every prime number produce non-prime or composite numbers.

Allowing for familiarity, this device for finding primes will benefit those students encountering primes for the first time and it will be a stepping stone for the others who can delve into such interesting facts as finding the highest number of decade primes (i.e. the number of primes from 20 to 30, etc.), comparing the number of primes in each worksheet, and finding the difference of two consecutive primes. Personal experiences with teachers and students have shown this minor number theorizing to be interesting and stimulating in learning mathematics and in strengthening needed computational skills for these students.

Prime Numbers: Worksheet 1

Rules

1. Cross out all multiples of 2 (nos. ending in 0, 2, 4, 6, 8,) excluding 2.
2. Cross out all multiples of 3 (nos. whose sum of digits is divisible by 3) excluding 3.
3. Cross out all multiples of 5 (nos. ending in 0 or 5) excluding 5.

Nos. from 1 to 100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of Primes Not Exceeding 100

17	11	13	
14	22	26	
21	33	39	17 / 34
28			
91			47 / 94

List Primes: 2,3,5,7, . . .

Families of Curves Inversely Related to the Conic Sections and Cassinian Ovals

by Rick Roesler
Lake Highlands High School
Dallas, Texas

Abstract

Many families of curves exhibit the property of eccentricity. In two families, the conic sections and the Cassinian ovals, the curves are defined as the ratio and product of two distances, respectively. By taking the inverse relationship between the two distances in each family, two new families of curves are generated, each of which exhibits eccentric properties.

Introduction

Eccentricity is a property of various families of curves (e.g. — the conic sections, the Cassinian ovals, the cycloids, the conchoid of Nicomedes, and the limacon of Pascal) obtained by fixing all parameters but two. Generally, in four-parameter curves, the center of the curve (or the vertex if it is not a central curve) is fixed. The eccentricity, then, is the quotient of the remaining two parameters.

In the conic sections and the Cassinian ovals, the curve is defined as the quotient and product of two distances, respectively. That the curve (or family of curves) obtained by taking the inverse relationship between the distances also exhibits eccentric properties suggests itself.

The Inverse Conic Sections

A conic section is the locus of a point P that moves in the plane of a fixed point F, the focus, and a fixed line d, the directrix, such that the ratio of the distance of P from F to its distance from d is a constant e, the eccentricity. Therefore, an inverse conic with center C(h,k) is defined as the locus of a point P which moves in the plane of a fixed point F(h,k+a), the focus, and a fixed line $y = k - a$, the directrix, such that the product of the distance of P from F and its distance from the directrix is a constant c^2:

$$\sqrt{(x-h)^2 + [y-(k+a)]^2} \cdot |y-(k-a)| = c^2 \qquad C \neq O \qquad (1)$$

For convenience in examining the properties of the family of curves, let C lie at the origin. Hence,

$$\sqrt{x^2 + (y-a)^2} \cdot |y+a| = c^2 \qquad (1a)$$

The eccentricity of the curve is the absolute value of the ration of a to c, where a is the directed distance from C to F:

$$e = \frac{|a|}{|c|}$$

The same curve may be described by a polar equation in which the pole is taken to be at the focus:

$$\rho = \frac{-a \pm \sqrt{a^2 + c^2 \operatorname{Sin} \Theta}}{\operatorname{Sin} \Theta} \quad \theta \neq 0, \pi \qquad (2)$$

$$\rho = \frac{c^2}{2a} = \frac{c}{2c} \qquad \theta = 0, \pi \qquad (2a)$$

From equation (1) it follows that the line $x=h$ is an axis of symmetry. To solve equation (2) for the points on this line, let $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$:

$$\rho = -a \pm \sqrt{a^2 + c^2} \qquad (3a)$$

$$.\wedge. \qquad \rho = a \pm \sqrt{a^2 - c^2} \qquad (3b)$$

Equation (3a) yields two real values for all values of a and c. Equation (3b), however, yields a number of real roots dependent upon the values of a and c:

Case I	: $a<c$	$(e<1)$	0 real roots
Case II	: $a=c$	$(e=1)$	1 real root
Case III	: $a>c$	$(e>1)$	2 real roots

Since there is no stipulation stating $a \neq 0$ (the focus cannot lie on the directrix), Case I may be subdivided into two parts:

Case Ia: $a=0$ $(e=0)$		0 real roots
Case Ib: $0<a<1$ $(0<e<1)$		0 real roots

Hence, combining the results obtained from equations (3a) and (3b), the number of points lying on the axis of symmetry for the four cases is

Case Ia	: $e=0$	2 points (inverse circle)
Case Ib	: $0<e<1$	2 points (inverse ellipse)
Case II	: $e=1$	3 points (inverse parabola)
Case III	: $e>1$	4 points (inverse hyperbola)

See figure 2.

The distance, s, of these points from C may be found from equations (3a) and (3b):

$$s = \sqrt{a^2+c^2} \quad .\wedge. \quad s = \sqrt{a^2-c^2}$$

Thus, the outermost points on the axis lie at a distance $\sqrt{a^2+c^2}$ from C, while the innermost point(s), if any, lie at a distance $\sqrt{a^2-c^2}$ from C. For the inverse circle, since $e=0$ $(a=0)$, the two outer points are at a distance c from C. For the inverse parabola, since $e=1$ $(a=c)$, the two outer points lie at a distance $a\sqrt{2}$ from C, while the inner point is coincident with C.

Figure 1. The Conic Sections $e = \frac{FP}{DP}$

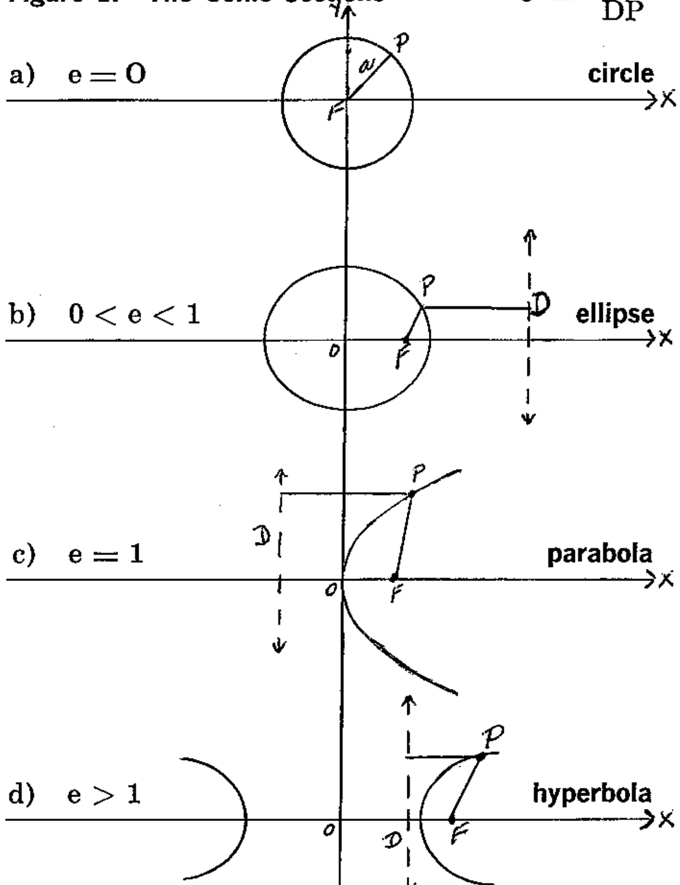
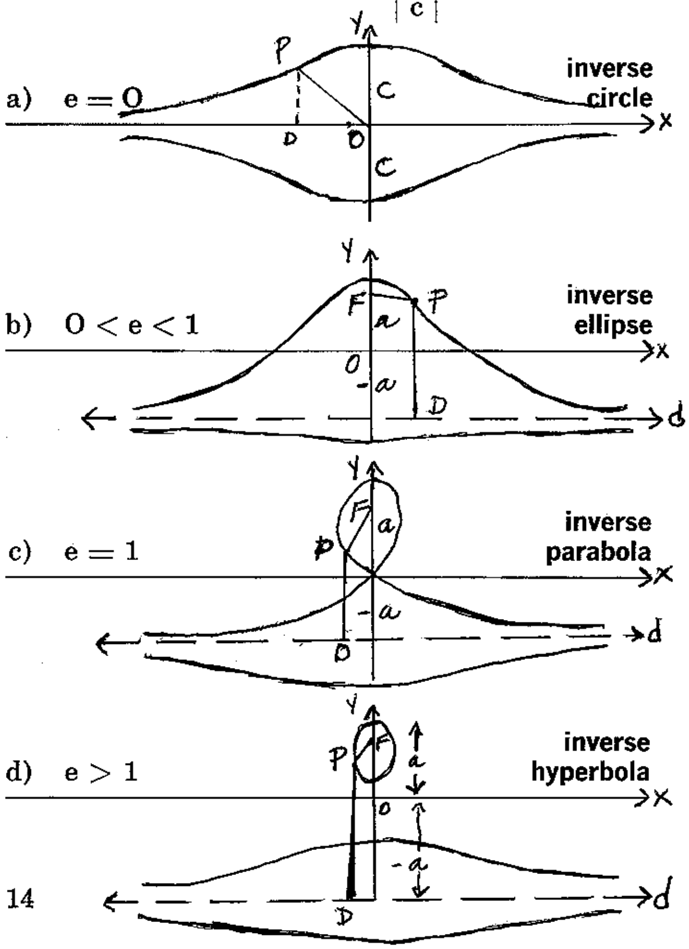


Figure 2. The Inverse Conic Sections

$FP \cdot DP = c^2 \qquad e = \frac{|a|}{|c|} \qquad a = OF$



Solving equation (1a) for x:

$$x = \pm\sqrt{\frac{c^4}{(y+a)^2} - (y-a)^2}$$

$$\lim_{y \to -a} x = \pm\infty$$

Hence, in each case the curve is asymptotic to the directrix, $y = -a$.

The Inverse Cassinian Ovals

A Cassinian oval is the locus of a point P in a plane such that the product of its distances from two fixed points in the plane, F and F′, is a constant, k^2. In a manner similar to that used for the inverse conics, an inverse Cassinian oval with center C(h,k) is the locus of a point P in a plane such that the ratio of its distances from two fixed points in the plane, F(h+b,k) and F′(h—b,k), is a constante e, the eccentricity of the curve, where b is the directed distance from C to F:

$$\frac{FP}{F'P} = \frac{\sqrt{[x-(h+b)]^2 + (y-k)^2}}{\sqrt{[x-(h-b)]^2 + (y-k)^2}} = e$$

For convenience in examining the properties of the curve, let C lie at the origin. Hence,

$$\frac{\sqrt{(x-b)^2 + y^2}}{\sqrt{(x+b)^2 + y^2}} = e$$

Simplifying equation (1),

$$\left(x - b\frac{1+e^2}{1-e^2}\right)^2 + y^2 = \frac{4b^2e^2}{(e^2-1)^2}$$

For the case where $e \neq 1$, this is the equation of a circle with center C′ $(b\frac{1+e^2}{1-e^2}, 0)$ and radius and radius $r = \frac{2be}{|e^2-1|}$. If $e = 1$, the locus is, by definition, the perpendicular bisector of the segment $\overline{FF'}$, since every point is equidistant from the two endpoints. If $e = 0$, the locus is a point circle at F. In the limit, as $e \to \infty$, the locus is a point circle at F′.

To examine the circles more closely, the relationship between the abscissa of C′ and the eccentricity may be obtained from a graph of the function $h' = b\frac{1+e^2}{1-e^2}$. A graph of the radial function $r = \frac{2be}{|e^2-1|}$ also provides useful information (see figures 4, 5).

Combining the results of the two graphs, where $O = \frac{1+\sqrt{5}}{2} \approx 1.618$, the golden number:

$$\lim_{e \to 0+} h' = b \qquad \lim_{e \to 0+} r = 0$$

$$\lim_{e \to 0-1} r = 2b$$

$$\lim_{e \to 1-} h' = \infty \qquad \lim_{e \to 1-} r = \infty$$

$$\lim_{e \to 1+} h' = -\infty \qquad \lim_{e \to 1+} r = \infty$$

$$\lim_{e \to 0} r = 2b$$

$$\lim_{e \to \infty} h' = -b \qquad \lim_{e \to \infty} r = 0$$

This provides enough information to graph the set of inverse Cassinian ovals (see figure 6).

Conclusions

By taking the inverse relationship between the two distances defining the conic sections and Cassinian ovals, two new families of curves are generated, the inverse conic sections and the inverse Cassinian ovals. Like the conics and the ovals, these families of curves exhibit the property of eccentricity. In the two families in which the curves are defined by a quotient relationship (the conics and inverse Cassinian ovals), the eccentricity is found to be equal to the ratio of the defining distances. In the two families in which the curves are defined by a product relationship (the Cassinian ovals and inverse conics), the eccentricity is equal to the ratio of the square root of the defining constant to the distance between the center and the focus.

Figure 3. Cassinian Ovals

$F'P \cdot FP = k^2 \quad e = \frac{|k|}{|b|} \quad b = OF$

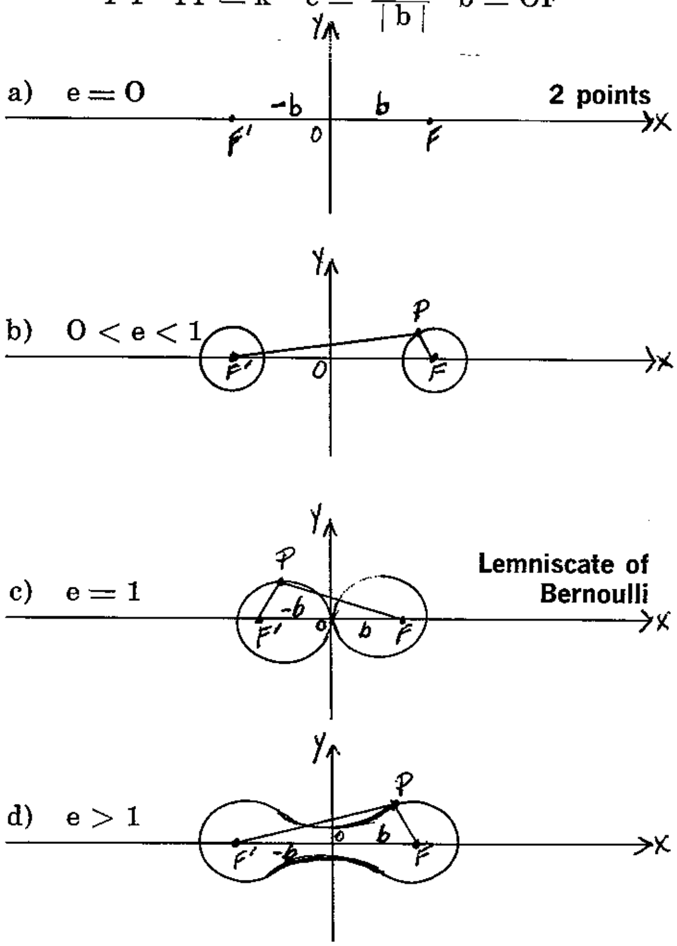


Figure 4. $h' = b\frac{1+e^2}{1-e^2}$

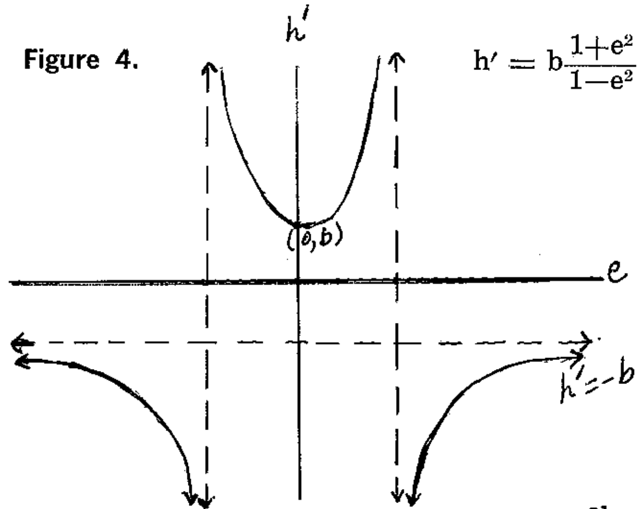


Figure 5. $r = \frac{2be}{[e^2-1]}$

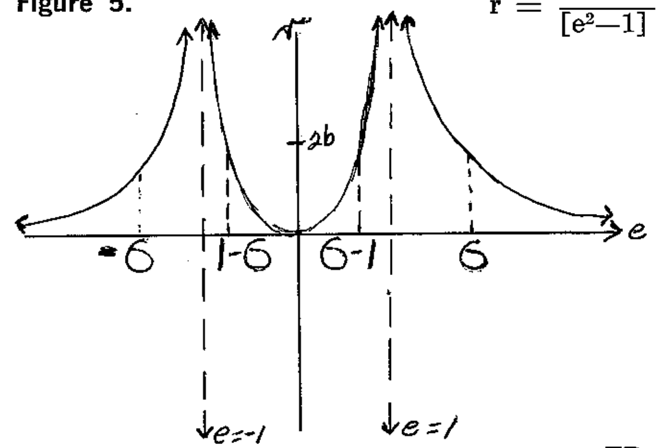
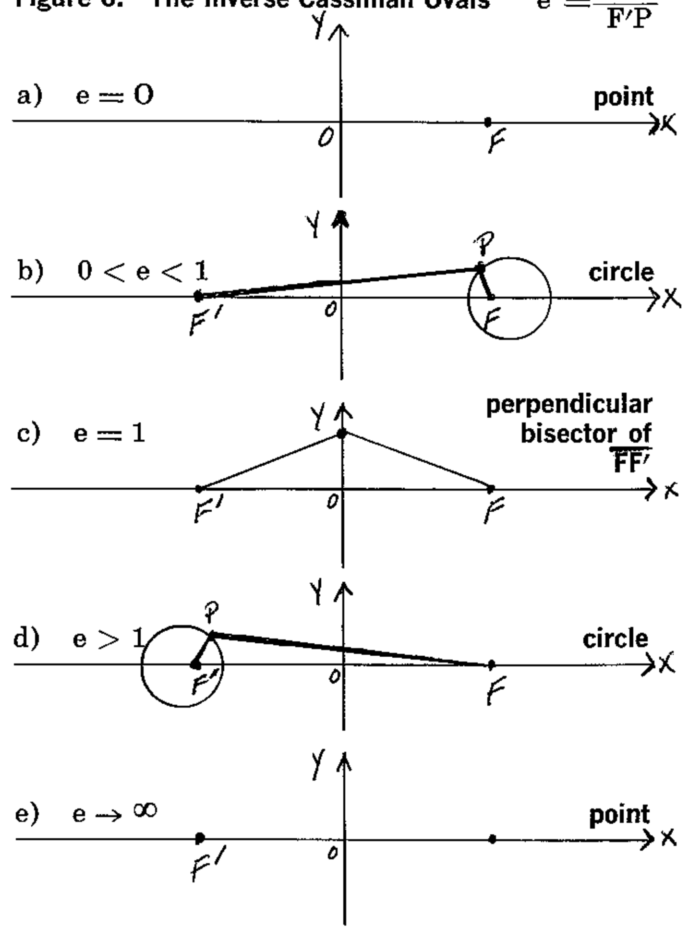


Figure 6. The Inverse Cassinian Ovals $e = \frac{FP}{F'P}$



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