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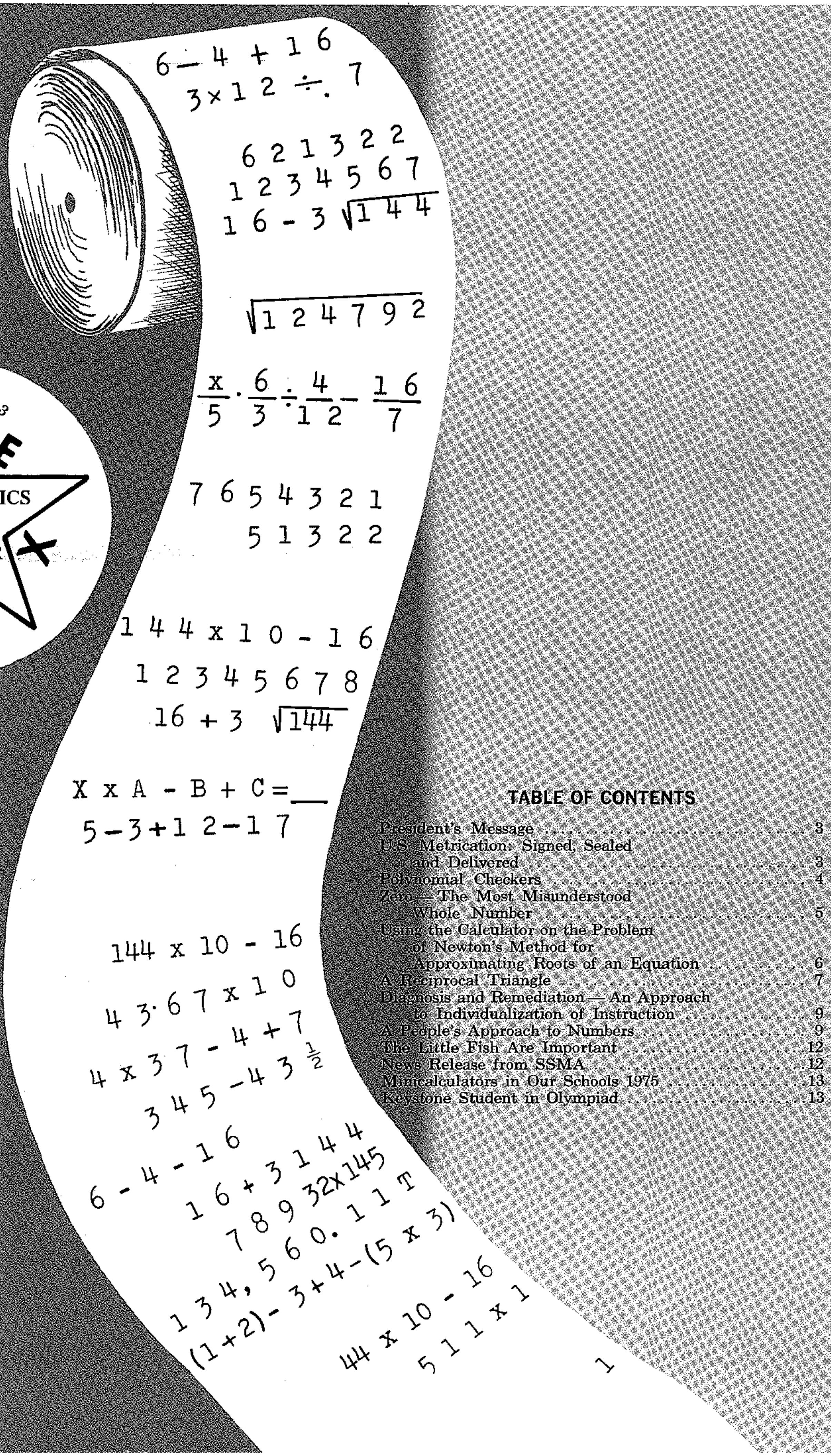

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President's Message

As we come to the end of another school year, perhaps we should pause to reflect on a few of the events of the year which might give us cause to feel we have shared in a job well done. To be sure, many things that ought to have been done have gone into the past, and some of our failures will haunt us in the future. Yet, I feel this year has had accomplishments worth noting.

Our CAMT meeting was successful beyond imagination, with 1101 registering—the largest number ever. Metrification became more nearly a reality with a bill being signed at Christmas-time, although no firm implementation date was included. Texas gained her first NCTM Director in recent years with the election of Betty Beaumont. For these things we can be proud.

At the NCTM convention last month in Atlanta, the degelates were seiged with resolutions asking for more financial assistance on the local level from NCTM. In view of these resolutions adding additional cost for membership to all of us, and only a few benefiting. the majority of delegates opposed such moves.

Shirley Cousins and I are working on the program for a fall workshop to be held on November 13 at Dobie High School. Barring conflicts with football games, please reserve this date for us and plan to attend.

In early August, you will receive a membership reminder. This will serve two purposes—to get membership in early before the rush of school begins and also to find membership changes of address. Should you be willing to assist us as a building membership leader, please notify me for membership forms.

By the way, one other noteworthy item from Atlanta. Some of our friends to the North presented a resolution to assign delegates on a membership basis. It was pointed out that Texas has 22 groups affiliated with NCTM and Northern neighbor had only 3. It is my feeling that the large number of local organizations is what makes our state as active. Personally, I am proud of each of our local affiliates and eagerly await the day when more are formed. This was not intended to be critical of the resolution. Rather, it was meant to inform you that many are envious of the work you do in our state. I am proud to be a part of you.

HAVE A GOOD SUMMER!

BILL ASHWORTH

U. S. Metrication: Signed, Sealed and Delivered?

by James Bezdek and George Willson
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The 23rd day of December, 1975, is destined to become a day of historical significance in the world of measurement. It was on this day that President Gerald Ford signed into law the Metric Conversion Act. Thus the United States became the last major country to officially sanction conversion to the metric system of measurment.

The policy, as defined in the Metric Conversion Act, shall be to coordinate and plan the increasing use of the metric system in this country. To implement this policy, the Act provides for the establishment of a 17-member United States Metric Board to coordinate a voluntary conversion to the metric system. President Ford indicated that the 10-year conversion plan will make the metric system the country's "predominant, but not exclusive" system.

With Federal Government backing, metrication in the United States is now "signed" into law. The concrete "sealing" of metrication is being supplied by business and industrial concerns which buy and sell products here and overseas. They have been a dominant force in hastening metrication in the United States. For a long time, producers of items ranging from those in the supermarket to the auto industry, have utilized metric measure without public alarm. Recent announcements, such as the one[1] proclaiming that the individual races in the Texas Relays this spring will be run in metres in preparation for the 1976 Olympics, are appearing in increasing numbers. As these increase, public attention and interest will be aroused. Thus it appears the "sealing" process is well underway and metrics is here to stay.

The third phrase of metrication important to educators is "delivery." The writers believe that educators in the United States will have the major responsibility for delivering full metrication. The voluntary implementation will not come about peacefully if the public, i.e., parents, students, and teachers, do not readily accept the responsibilities associated with a major changeover to another system of measurement.

Who is responsible for delivery? We believe educators should provide the leadership and direction in metrication so that today's pupils, as future adults, will be able to use the metric system in their day-to-day living. Schools should be and are beginning to teach the metric system to their students.

Therefore, Education in the United States is faced with major questions and decisions regarding implementation in the classroom. We need to know the status of knowledge that pre-service teachers, in-service teachers, and the general public have in the metric system; how this knowledge can be measured; and what kind of preparative programs are needed for these teachers and the general public.

For efficient and effective planning of any program of preparation for a specific segment of our population, it becomes obvious that there must be information available regarding their knowledge of the subject. Upon reviewing this situation, it was found that it is not really known what the levels of understanding are for the various sectors. Further review also revealed that there was not any way of determining the levels of knowledge due to the nonexistence of a measuring device. Therefore, if work was to be done in this area, the development of a measuring device for the area of metrics was the first priority. The two writers undertook a research project, the major objective being to develop an instrument for measuring metric knowledge.

The instrument[2] developed consists of two forms — A and B — which measure basic knowledge of the metric system. Each form consists of 20 multiple-choice items. These include items dealing with the basic units of measure, the prefixes and the number associated with each one, application of the basic units to real-life situations, and conversions within the metric system. A pilot group of 188 subjects was used in standardization, which included an item analysis and test of reliability and validity. The test-retest procedure to determine reliability produced a .94 coefficient.

Since the writers have a vested interest in the training of pre-service teachers, the tests were then used with 177 pre-service teachers (senior elementary education majors). An analysis of the scores obtained produced a mean of 11.03 with a standard deviation of 3.49. Even though the sample is relatively small and not representative of all pre-service teachers, it does reveal some interesting results if applicable to a larger sample. Obviously, efforts must be undertaken to place more emphasis on this area of study in the training of teachers. (To further verify the status of pre-service teachers' knowledge, the writers are in the process of collecting data covering a large sample of pre-service teachers in five states.)

If the above results are a reasonably accurate assessment of pre-service teachers' knowledge, then what about in-service teachers and the general public? Not only should we be concerned about the status of their basic knowledge, but we should also be concerned about the preparation of these groups and the length of their transition period. (Ten years as suggested by the Metrication Act?)

The writers are continuing research studies to provide a basis for answers to these questions. One study underway will determine the status of knowledge of metrics of in-service teachers, and another is concerned with the metric knowledge of "John Q. Public."

If the predicted results are verified, then Education in the United States has its work cut out. It appears metrication in this country is "signed and sealed." It is now up to us as educators to see that full metrication is "delivered."

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Polynomial Checkers

By
H. E. Bible and G. K. Goff
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Historically the best set of arithmetic checkers included "casting out nines." These techniques could be used to check additions, subtractions, multiplications and divisions in arithmetic.

Since any positive integer can be represented as a polynomial over the integers with the variable equal to 10, then casting out nines will work for polynomials using only the coefficients.

$$(12x^3 + 3x^2 - 5) + (x^2 - 2x + 3) = 12x^3 + 4x^2 - 2x - 2$$

$$(12 + 3 - 5) + (1 - 2 + 3) \qquad 12 + 4 - 2 - 2$$

$$(9 + 1) + 2 \qquad 9 + 3$$

$$3 \qquad 3$$

$$(12x^3 + 3x^2 - 5) - (x^2 - 2x + 3) = 12x^3 + 2x^2 + 2x - 8$$

$$(12 + 3 - 5) - (1 - 2 + 3) \qquad 12 + 2 + 2 - 8$$

$$10 - 2 \qquad 8$$

$$(12x^3 + 3x^2 - 5)(x^2 - 2x + 3) = 12x^5 - 21x^4 + 30x^3 + 4x^2 + 10x - 15$$

$$(12 + 3 - 5)(1 - 2 + 3) \qquad 12 - 21 + 30 + 4 + 10 - 15$$

$$(10)\ (2) \qquad 3 - 3 + 3 + 4 + 1 - 6$$

$$1 \cdot 2 \qquad 2$$

$$(12x^3 + 3x^2 - 5) \div (x^2 - 2x + 3) = 12x + 27 + \frac{18x - 86}{x^2 - 2x + 3}$$

$$(12 + 3 - 5) \div (1 - 2 + 3) \qquad 12 + 27 + \frac{-68}{2}$$

$$10 \div 2 \qquad 39 - 34$$

$$5 \qquad 5$$

However, observing the examples just given, casting out nines was not necessary since true equality exists in the first step of each example. The next question of course is; "will this always work?" Being suspicious that the answer is yes the next step is to try and prove it.

Let $P(x) - a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, and $Q(x) = b_m x^m + b_{m-1} x^{m-1}\, x \cdots b_1 x + b_0$ be polynomials over the integers.

Then $P(x) + Q(x) = \sum_{i=o}^{\max(m,n)} (a_i + b_i)\, x_i$; $a_k = 0$, $k > n$ and $b_h = 0$, $h > m$. Since $\sum_{i=o}^{\max(m,n)} (a_i + b_i) = \sum_{i=o}^{n} a_i + \sum_{i=o}^{m} b_i$

The check works for addition of polynomials. It is also obviously true for subtraction of polynomials.

The case for multiplication appears to be somewhat more formidable since,

$$P(x) \bullet Q(x) = \sum_{i=o}^{m+n} c_i x^i \text{ where } c_i = \sum_{j=o}^{i} a_{i-j} b_j$$

$i = 0, 1, \cdots, m + n$ and $a_k = 0$ if $k > n$ and $b_h = 0$ if $h > m$. The proof for multiplication is possible by taking an arbitrary *m* and inducting on *n*.

Since division is not closed in polynomials, the proof for division appears to be even more formidable. So, we suppose now would be a good time to expose a nice simple proof for all cases.

In polynomials, each of the operations must give the same result for all values of x in its domain, and most self respecting domains contain a multiplicative identity, so let $x = 1$ and the proofs result.

One useful application of the check for multiplication would be when your "deliberate abstracter" insists that $(x + 3) \cdot (x + 2) = x^2 + 6$. Using the check he should become suspicious of his answer since $(1 + 3)\ (1 + 2) \neq 1 + 6$ and "12 ain't 7."

Zero—The Most Misunderstood Whole Number

By Steven Hatfield
Marshall University

After teaching freshman mathematics on the college level for thirteen years, I have observed that the following properties of zero have given many of my students problems. These problems are rooted in the very early stages of the students' mathematical training.

In elementary school zero is defined to be the number property of the empty set. Special emphasis is needed to clarify the point that zero is not included in the null set.

—0

The student first encounters —0 when he considers the solution of problems of the type 7 — 0 = □. After he is able to find □, hopefully he will remember that 7 + 0 = 7 and then be able to recognize the relationship between —0 and +0.

Multiplication by 0

At present there are three definitions of multiplication being used in most elementary school textbooks: set, array, and cross-product definitions. I found that my freshmen are able to understand why $7 \bullet 0 = 0$ by using any of these definitions.

Division by 0

The following is a hypothetical exchange involving a student and a teacher:

Teacher: What is zero divided into zero?

Student: Zero.

Teacher: Why not one?

Student: Beacause $0 \times 0 = 0$.

Teacher: Well, 0×1 equals 0 doesn't it? If you do not think the answer is 1, what about 74?

Obviously this confusion could be alleviated if the teacher stressed the three cases of division by zero. Division involving zero is usually understood by reference to multiplication. Thus if $y = 0$ $0/y = 0$ because $0 \times y = 0$. Now consider $y/0 = N$ can be written $N \times 0 = y$. Here N stands for no number. The other case where 0 is both divisor and dividend, $0/0 = N$ can be written $N \times 0 = 0$. Here N represents any number which is impossible because the result of an operation must be a unique number. The use of a small calculator is a good method to illustrate the different cases, especially when a student is having difficulty understanding the various concepts.

X^0

In Algebra I or earlier, a student is confronted with problems of the type $\frac{2^2}{2^2}$ or $\frac{b^2}{b^2}$, At this time the concept of x^0 should be introduced not as it is in many books weeks or as long as a year later. This would make the definition more meaningful for the student and help the student understand why x^0 is not defined. (Of course, here we assume the student understands division by 0.)

0!

In Algebra II, the student sometimes encounters the concept of binominal coefficients. Recalling the definition of "N things taken N at a time" will help a student see why 0! is defined to be one.

Using the Calculator on the Problem of Newton's Method for Approximating Roots of an Equation

By Sterling C. Crim

Lamar University

In the October '75' issue of the Texas Mathematics Teacher, Mr. Reeves stated his goals for calculator use in the classroom, one which is appropriate to the problem mentioned in the title of this article and I will quote the goal: iii) "to eliminate computational mistakes (thus reducing frustration) that plague some students when they attack word problems that they know how to solve."

In the pre-calculator days the student "could not see the forest for the trees" in the problem of approximating the roots of an equation. Fall of '75' was the first semester that all of the students in the class of Calculus I that I taught had access to a calculator. As a result of total student use of the calculator, the homework on Newton's Method" had a variety of exercises where as past assignments were limited to two problems.

The following is an example from the Third Edition of Thomas' Calculus and a program for the H. P. — 45 Solution.

Problem: Estimate the root of the equation

$x^4 + x - 3 = 0$ between 1 and 2

Solution:
$$f(x) = x^4 + x - 3$$
$$f(1) = 1^4 + 1 - 3$$
$$= -1$$
$$f(2) = 2^4 + 2 - 3$$
$$= 16 + 2 - 3$$
$$= 15$$

f(x) is a continuous function

$f(1) = -1$ and $f(2) = 15$ shows that the graph $y = f(x)$ crosses the x-axis between 1 and 2.

Using the recursive formula for Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = x^4 + x - 3$$
$$f'(x) = 4x3 + 1$$

$$\text{Let } x_1 = \frac{1 + 2}{2}$$
$$= 1.5$$

PRESS

x_n [ENTER↑] [STO] 1 [x^2] [x^2] [STO] 2
[RCL] 1 [ENTER↑] [RCL] 2
[+] [ENTER↑] 3 [−] [STO] 3
[RCL] 1 [x^2] [ENTER↑] [RCL] 1 [×]
[ENTER↑] 4 [×] [ENTER↑] 1 [+]
[ENTER↑] [RCL] 3 [X ≶Y] [÷] [ENTER↑]
[RCL] 1 [X ≶Y] [−]

Display is x_{n+1}, record and let $x_n = x_{n+1}$ and start program over.

$x_1 = 1.5$

$x_2 = 1.254310345$

$x_3 = 1.172277657$

$x_4 = 1.164110042$

$x_5 = 1.164035146$

$x_6 = 1.164035140$

$x_7 = 1.164035140$

For once the groans that usually accompany this problem are missing.

A Reciprocal Triangle

By Bonnie H. Litwiller and David R. Duncan
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Most mathematics teachers have encountered Pascal's Triangle as shown in Figure I.

Figure I: Pascal's Triangle

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
    1   6  15  20  15   6   1
  1   7  21  35  35  21   7   1
1   8  28  56  70  56  28   8   1
                .
                .
                .
```

There are two equivalent methods of generating this triangle:

1. Addition Method

Beginning with the two exterior diagonals of ones, generate the interior entries by adding each pair of consecutive numbers in the same row and recording the sum directly beneath and between the entries added. For example, the second and third entries of the fifth row are respectively 4 and 6; since $4 + 6 = 10$, 10 is recorded beneath and between 4 and 6 as the third entry in the sixth row.

If $P(i,j)$ is used to denote the jth entry of the ith row, the previous example may be written symbolically as $P(5,2) + P(5,3) = P(6,3)$. In general, the defining relation for generating Pascal's Triangle by the addition method is $P(i,j) + P(i,j+1) = P(i+1,j+1)$.

2. Combinatorial method

An important idea in probability concerns the number of subsets of size j which can be selected from an original set of size i. This number, denoted as $\left(\frac{i}{j}\right)$ can be shown to equal $\frac{i!}{j!(i-j)!}$. It can also be shown that $P(i,j)$ is in fact $\left(\frac{i-1}{j-1}\right)$. For example,

$$P(7,5) = \left(\frac{6}{4}\right) = \frac{6!}{4!2!} = \frac{6 \cdot 5}{2 \cdot 1} = 15.$$

Now consider a less familiar triangle as shown in Figure II. Call this triangle the Reciprocal Triangle.

Figure II: The Reciprocal Triangle

$$\frac{1}{1}$$
$$\frac{1}{2} \quad \frac{1}{2}$$
$$\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3}$$
$$\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4}$$
$$\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5}$$
$$\frac{1}{6} \quad \frac{1}{30} \quad \frac{1}{60} \quad \frac{1}{60} \quad \frac{1}{30} \quad \frac{1}{6}$$
$$\frac{1}{7} \quad \frac{1}{42} \quad \frac{1}{105} \quad \frac{1}{140} \quad \frac{1}{105} \quad \frac{1}{42} \quad \frac{1}{7}$$
$$\vdots$$

Build this triangle by writing in the left exterior diagonal the reciprocals of the natural numbers: $\frac{1}{1'}\frac{2}{1'}\frac{3}{1'}\frac{4}{1'}$. . . Each other entry is then completed by subtracting the entry in the same row immediately to its left from the number directly above in the preceding row and between them. For example, the second entries of each row (beginning with row 2) can be expressed as

$$\frac{1}{2} = \frac{1}{1} - \frac{1}{2}\,;\ \frac{1}{6} = \frac{1}{2} - \frac{1}{3}\,;\ \frac{1}{12} = \frac{1}{3} - \frac{1}{4}\,;\ \frac{1}{20} = \frac{1}{4} - \frac{1}{5}\,;\ \ldots$$

The third entries of each row beginning with row 3 can be expressed as

$$\frac{1}{3} = \frac{1}{2} - \frac{1}{6}\,;\ \frac{1}{12} = \frac{1}{6} - \frac{1}{12}\,;\ \frac{1}{30} = \frac{1}{12} - \frac{1}{20}\,;\ \frac{1}{60} = \frac{1}{20} - \frac{1}{30}\,;\ \ldots$$

If $R(i,j)$ is used to denote the jth entry of the ith row of the Reciprocal Triangle, the defining relation becomes $R(i,j) - R(i+1,j) = R(i+1,j+1)$.

Using Figure III as a reference, we note several number patterns.

Figure III

Diagonal I, Diagonal II, Diagonal III, Diagonal IV

$$\frac{1}{1}$$
$$\frac{1}{2} \quad \frac{1}{2}$$
$$\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{3}$$
$$\frac{1}{4} \quad \frac{1}{12} \quad \frac{1}{12} \quad \frac{1}{4}$$
$$\frac{1}{5} \quad \frac{1}{20} \quad \frac{1}{30} \quad \frac{1}{20} \quad \frac{1}{5}$$
$$\frac{1}{6} \quad \frac{1}{30} \quad \frac{1}{60} \quad \frac{1}{60} \quad \frac{1}{30} \quad \frac{1}{6}$$
$$\frac{1}{7} \quad \frac{1}{42} \quad \frac{1}{105} \quad \frac{1}{140} \quad \frac{1}{105} \quad \frac{1}{42} \quad \frac{1}{7}$$
$$\vdots$$

1. Consider the denominators of the diagonals.

In Diagonal I, the denominators are 1, 2, 3, 4, 5,

. . . or the set of natural numbers. The natural numbers occur in Diagonal II of Pascal's Triangle.

In Diagonal II, the denominators are 2, 6, 12 20, 30 42, . . . Rewrite this set as 2 • 1, 2 • 3, 2 • 6, 2 • 10, 2 • 15, 2 • 21, . . . and note that these are two times the triangular numbers. The triangular numbers appear in Diagonal III of Pascal's Triangle.

In Diagonal III, the denominators are 3, 12, 30, 60, 105, . . . Rewrite this set as 3 • 1, 3 • 4, 3 • 10, 3• 20, 3 • 35, . . . and notice that these are three times the tetrahedral numbers. The tetrahedral numbers appear in Diagonal IV of Pascal's Triangle.

In Diagonal IV, the denominators are 4, 20, 60, 140, . . . Rewrite this set as 4 • 1, 4 • 5, 4 • 15, 4 • 35, . . . and note that these are 4 times the numbers which lie in Diagonal V of Pascal's Triangle.

In general, the denominators of the entries of the *q*th diagonal of the Reciprocal Triangle are equal to q times the entries of the (q + 1) diagonal of Pascal's Triangle.

2. Consider the differences between consecutive denominators in each diagonal of the Reciprocal Triangle.

Diagonal I:

1 2 3 4 5 6 7 . . .

1 1 1 1 1 1 . . .

Note the differences are all 1.

Diagonal II:

2 6 12 20 30 42 . . .

4 6 8 10 12 . . .

(2·2) (2·3) (2·4) (2·5) (2·6) . . .

Note the differences are two times the natural numbers (starting with the second natural number).

Diagonal III:

3 12 30 60 105 . . .

9 18 30 45 . . .

(3·3) (3·6) (3·10) (3·15) . . .

Note the differences are three times the triangular numbers (starting with the second triangular number).

Diagonal IV:

4 20 60 140 . . .

16 40 80 . . .

(4·4) (4·10) (4·20) . . .

Note the differences are 4 times the tetrahedral numbers (starting with the second tetrahedral number).

3. The product of any two consecutive entries in Diagonal I represents the next to last entry in the row containing the second number used as a factor.

For example, $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$; $\frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$; $\frac{1}{4} \cdot \frac{1}{5} = \frac{1}{20}$; . . . .

4. Consider the Reciprocal Triangle written in factored form; see Figure IV.

Figure IV

$$\frac{1}{1\cdot 1}$$

$$\frac{1}{2\cdot 1} \quad \frac{1}{2\cdot 1}$$

$$\frac{1}{3\cdot 1} \quad \frac{1}{3\cdot 2} \quad \frac{1}{3\cdot 1}$$

$$\frac{1}{4\cdot 1} \quad \frac{1}{4\cdot 3} \quad \frac{1}{4\cdot 3} \quad \frac{1}{4\cdot 1}$$

$$\frac{1}{5\cdot 1} \quad \frac{1}{5\cdot 4} \quad \frac{1}{5\cdot 6} \quad \frac{1}{5\cdot 4} \quad \frac{1}{5\cdot 1}$$

$$\frac{1}{6\cdot 1} \quad \frac{1}{6\cdot 5} \quad \frac{1}{6\cdot 10} \quad \frac{1}{6\cdot 10} \quad \frac{1}{6\cdot 5} \quad \frac{1}{6\cdot 1}$$

$$\vdots$$

In general it appears that $R(i,j) = \frac{1}{iP(i,j)}$. Let us verify this conjecture. First, let us verify that the recursive formula $R(i,j) - R(i+1,j) = R(i+1,j+1)$ is satisfied using the value of R(i,j) conjectured above. That is, it must be shown that

$$\frac{1}{iP(i,j)} - \frac{1}{(i+1)P(i+1,j)} = \frac{1}{(i+1)P(i+1,j+1)}.$$

But

$$\frac{1}{iP(i,j)} - \frac{1}{(i+1)P(i+1,j)} = \frac{1}{i\binom{i-1}{j-1}} - \frac{1}{(i+1)\binom{i}{j-1}}$$

$$= \frac{1}{i\frac{(i-1)!}{(j-1)!(i-j)!}} - \frac{1}{(i+1)\frac{(i)!}{(j-1)!(i-j+1)!}}$$

$$= \frac{(j-1)!(i-j)!}{i!} - \frac{(j-1)!(i-j+1)!}{(i+1)!}$$

$$= \frac{(i+1)(j-1)!(i-j)! - (j-1)!(i-j+1)!}{(i+1)!}$$

$$= \frac{(j-1)!(i-j)![(i+1) - (i-j+1)]}{(i+1)!}$$

$$= \frac{(j-1)!(i-j)!\,j}{(i+1)!} = \frac{j!(i-j)!}{(i+1)!}$$

$$= \frac{j!(i-j)!}{(i+1)\,i!} = \frac{1}{(i+1)\frac{i!}{j!(i-j)!}}$$

$$= \frac{1}{(i+1)\binom{i}{j}} = \frac{1}{(i+1)P(i+1,j+1)} \text{ as desired.}$$

Since $\frac{1}{i} = R(i, 1)$ and $\frac{1}{i(1)} = \frac{1}{iP(i,1)}$ the formula $R(i,j) = \frac{1}{iP(i,j)}$ yields the correct results for all the entries along the left diagonal. The formula thus yields the entire Reciprocal Triangle.

Teachers and their students are invited both to verify the other number patterns which were observed and to seek and verify other patterns which may be found in the Reciprocal Triangle.

Diagnosis and Remediation—An Approach to Individualization of Instruction

By Charles E. Lamb
The University of Texas at Austin

It seems reasonable for the elementary classroom teacher of mathematics to organize some part of his (her) overall instructional program around diagnostic and remedial activities, especially those diagnostic and remedial activities concerned with computational errors. The consideration of computational errors has historically been an effective resource for teachers to use during instructional planning (Buswell, 1926 and Brownell and Hendrickson, 1950). It is my contention that diagnosis and remediation should be considered as a tool for further individualizing instruction in the classroom. As a model for individualized instruction, I have selected the framework developed by Trafton (1972).

The first level of organization to be considered is that of the group developmental approach. Teachers should take advantage of entire class situations to diagnose general sources of confusion. This may be done by paper and pencil testing, guided discussion, or by using open-ended questions, just to name a few things. In particular, where children are learning to use the basic operations of arithmetic, it is reasonable to suspect that an entire class might be having the same types of problems in computation. The key to this level of instruction is for the teacher to be flexible enough in method to allow for individual differences in children to appear.

The second level of organization more nearly approximates the traditional view of individualized instruction. It allows for individual progress and a flexible, small group approach to class organization. This structure of the classroom will lend itself easily to diagnostic and remedial activities. For example, by grouping children in small sets, children who are having a particular problem can receive help at the same time. Similarly, by keeping the groups flexible, children could receive extra help on a variety of problems in a short span of class time. This approach would allow each child to develop at his own pace by moving from group to group as the situation dictates. Also, children who are not having specific problems at any one time could serve as instructors for some of the small groups.

The third and final level of organization is that which allows children to select their own activities and self-pace themselves. After having diagnosed children's problem areas, it is the responsibility of the teacher to provide a selection of activities which will eliminate the errors. In this way, children will be motivated by choosing activities they like, and also they will be receiving vital reinforcement of basic and essential concepts in mathematics.

By using this framework, the important idea of diagnostic and remedial teaching may be implemented at all levels of classroom instruction and thus aid in the individualization process (Lamb, 1974).

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A People's Approach to Numbers

By Paul B. Johnson
University of California, Los Angeles

In our classes in the structure of arithmetic we tend to introduce sets, counting numbers based on sets, then generalize to whole numbers, fractional numbers, integers, rational numbers, reals and complex in approximately that order. On the basis of our superior knowlege we impose this structure and sequence on our students. They accept the structure as God given, like the air they breathe. Similarly, once having been told about it, they give it no more attention. Of course if things get smoggy they complain that it shouldn't happen to them.

An approach to encourage the student to impose his own structure has been used recently in a class for prospective elementary teachers. The aim is to help the student realize that structure is natural, and that people make such classifications in order to simplify their lives. Then, having made his own structure, he can better appreciate the structure urged by mathematicians. A sobering kick back is the possibility that while mathematicians are convinced the patterns we teach are inherent in the numbers themselves, they may not be the patterns which are most natural to the students in a particular class. If this is the case it may help the teaching process if the teacher realizes that what seems reasonable and obvious to him may be awkward, contrived and unnecessary to the instincts of the student.

The class begins with the instructor saying "as college students you are a class of well educated people. You already know a lot about numbers." (A wave of horror rolls over the class. They are well educated, yes. But half their education has been in academic subjects, and half on how to avoid mathematics. However, under non-threatening coaxing several will admit, now that its been mentioned, that they have heard the word 'number' before. The instructor continues.)

"Just to be sure we are all talking about the same thing, give me a sentence with a number in it and I'll write it on the board." To make this seem a reasonable thing to do, the instructor may ask a student to explain what he means by 'yellow' or some other convenient color name. The student, trying to explain in words, will mumble some nonsense (at least the instructor doesn't understand it). Eventually the student points out several yellow objects. Hurrah! The instructor may tease, and if the student has pointed to yellow ties, will say "I understand now. 'Yellow' is another word for 'tie'." Thus he shows that to explain a concept one gives examples showing the concept.

Suppose the students give the following sentences.

1. She mailed 37 Christmas cards.
2. Pat's hair is 19 inches long.
3. A baseball team has 9 players.
4. The picture is on page 42.
5. She purchased 4 pounds of potatoes.
6. Robert was a member of Company 598.
7. You triple your money if your bet wins.
8. Five eggs went into the omelet.
9. The lake holds about 4,000,000 gallons of water.
10. The president's room is on the third floor.
11. Half the married people are women.
12. Mary told the men to call her at 482-1134.
13. What number comes after 79?
14. The teacher added 6 points to Gene's score.
15. Treating the metal makes it four times as strong.
16. $4 + 7 = 11$.
17. Number 18 played in left field.
18. Sandra drank 6 quarts of orange juice last week.

With a little discussion all agree that each sentence shows a reasonable use of a number idea. The instructor then says "These 18 ideas are all numbers, yet are all different. Still, are some of them more alike than others? Do you feel that some of the ideas are of the same type and others are different? For example, is there any other number of the same type as in Sentence 1?"

Now the fun and thinking begins. Many students who have been given only very precise instructions in the past want more detailed instructions. Questions come up like "What do you mean, 'type'?" "How can we tell what to do unless you tell us exactly how two different numbers can be alike?" Some one might ask "Do you mean if we called the number in Sentence #1 yellow, then what are some of other yellow numbers?" That person has an A cinched, even if she wrote $9 \times 2 = 92$ on her multiple choice I.Q. test.

Some one might say that 1, 2, 3, are alike but different from 7, 8, 10, because digits are used in the first group and english spelling is used for the second group. The instructor may ask, "Are we sorting on number ideas?" Hopefully the class soon agrees that while this is a good way of sorting, it is not sorting by ideas but by a way of spelling the names. This is not what is wanted now.

Notice, the instructor has not told them to sort. The instructor has not characterized the sets he expects the students to sort into. He may even say that students and faculty often have valid different ideas about things (cheers from the back row) and he wants to see if student really think there are different types of numbers, and, if so how these differences are seen.

Soon someone may say something like "#1 and #3 are alike and different from #17." Without asking why, the instructor asks if others agree with this. He may ask "Let's list the ways we could classify the number ideas in #1, #2, #17.

The idea is of the same type in all 3.
{1,3,17}

Alike are #1 and #3 but #17 is different.
{1,3}, {17}

Alike are #1 and #17 but #3 is different.
{1,17}, {3}

Alike are #3 and #17 but #1 is different.
{3,17}, {1}

All three are different.
{1}, {3}, {17}

"Each of you has his own idea about which of these five statements is correct. Different people may very well have different ideas. Let's vote." The instructor then tallies the votes on a show of hands, being very careful not to show his opinion as he moves from possibility to possibility. Accuracy in counting is not important. The instructor by his attitude indicates he is not interested in individual answers. This is a straw ballot, the student is voting a feeling to which she is not necessarily committed, which she does not have to justify. If only one person votes for a possibility, the instructor may miscount and record 2 or 3 so the individual won't feel alone.

If the vote is unanimous, the instructor says "We all agree. I wonder if we all agreed for the same reasons" and calls for individuals to explain their vote. If there is a scattering of votes, he says 'we seem to differ. I wonder why" and calls for some one to explain the most popular opinion, and perhaps one or two others. At any time he may call for a revote. Thus opinions change and become clearer. Hopefully opinion $\{1,3\}$, $\{17\}$ will prevail because the numbers in #1 and #3 tell "how many" separate, countable objects are in a set and in #17 merely names a person.

The classification goes on, with various classifications made. For example $S = \{1,3,8\}$ might be one set because the numbers in these sentences tell how many are in a set. $G = \{2,5\}$ is a set because they describe how much of some continuous quantity there is, like the length of a segment. $Q = \{4,10\}$ might be a set because the sequential ideas are being used. There is no question of how many or how much, but merely the order. $M = \{6,12\}$ because here numbers are used only as names of something. $P = \{7,11\}$ because the numbers are used as operators, describing something which will happen to some other number.

There may well be differences of opinion. For example, is #18 a segment or continuous type number similar to those in G, or is #18 a set type number because orange juice comes in quart sized bottles and the student sees a set of 6 bottles? This shows the classification depends on the attitude of the classifier as well as the number properties themselves. There may be several correct classifications of any sentence.

Some student will complain that it isn't the numbers which are classified, it is the uses of the number. This student is congratulated and it is pointed out that classifying things by their uses is the modern scientific way of thinking. Such classifications are called "operational definitions" in physics. It is the method used in social science when people define anything as money which can be used the way non-numismatic coins are used.

The instructor may point out that we may want to develop classifications of numbers which are so widely accepted that they appear independent of the classifier. We don't fully succeed. For example "six" may be thought of as a whole number or a real number depending on the uses the classifier has in mind at the time. Mathematicians say "The whole numbers are imbedded in the real numbers."

We recognize that set S leads to the idea of cardinal number, G to real numbers, Q to ordinal, M to monimal, P to operators. All these ideas are widely used but often not linked together.

It is possible to classify "six" according to the way "six" is described rather than used. Mathematicians like to do this to show the number system can be described independently of use. This fairly narrow purpose does not seem to be as interesting or insightful as the "use" approach. Nor is it as complete, since it says that numbers on athletes are not really numbers because they can't be added. Nor is it as helpful in discussing relevancy, or, why should young children spend years of their lives studying this structure rather than some other equally intriguing one.

Several good things came from this approach. Students see that mathematics comes from their lives. Its content and structure are chosen to make life easier.

Numbers do not have to be based on sets. This is merely one approach taken to organize our thinking about them. It is simpler than other approaches in many cases but not in all. Other approaches are also important. Different people may well have different correct answers to problems as stated. This is especially true with problems close to life.

The author classified the numbers in the sentences as follows using the sets introduced above. The reader may not agree, for reasons just as good and maybe better than the author's.

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1S, 2G, 3S, 4Q, 5G, 6M, 7P, 8S, 9G, 10Q, 11P, 12M, 13Q, 14Q, 15P, 16X, 17M, 18G. X is a new classification for pure symbols with no use at all indicated.

North Texas CTM: 7th Annual Spring Banquet

The North Texas Council of Teachers of Mathematics held its seventh annual spring banquet at the Ramada Inn, Denton, Texas, April 1, 1976. N. A. Waters, Jr., gave the invocation. Jim Bezdek, president, presided. Entertainment was provided by Billie Nunley, Rose Marie Smith, and Bob Nunley. Reed Jackson, president-elect, introduced the speaker — Dr. Zalman Usiskin, University of Chicago. Dr. Usiskin gave a very interesting talk on "The Teaching of Geometry"—topics from all types of textbooks at all levels.

The NTCTM reported a membership of 127 members in 14 counties. Three were in attendance from Wichita Falls. Ernest Hrany, Plano, was introduced as the NCTM Representative.

The Little Fish Are Important

An Exercise In Logic

By Buck Martin
Bowling Green University

A deck of regular playing cards can be an effective teaching aid for elementary students as well as pre-schoolers. A variety of mathematical skills and concepts can be taught through such games as Blackjack, Casino, Cribbage, Go Fish, Poker and Rummy. A favorite card game in our house is War. In this game all cards except aces and deuces assume their normal face value. Aces outrank kings and deuces outrank aces. When we're lucky enough to find a deck with a joker in it, the joker outranks all other cards. Any number of people can play, but the game is best suited for two, three or four people. After all cards are dealt, each player exposes his top card. The player with the highest ranking card wins all exposed cards. Then the second cards are exposed, etc. In the event there is a tie for highest ranking card, a "war" occurs. This is resolved by having each player involved in the war deal three cards face down and a fourth card up. The player with the higher ranking up card wins all cards involved. In the event of another tie, the process is continued until the war is resolved. The game is usually played until one or more players lose all their cards. The player with the most cards at the end wins the game. (Note: Many teachers have effectively used classroom variations of this game involving number facts or comparison of fractions, e.g. 6 x 8 beats 5 x 9 or $\frac{3}{4}$ beats $\frac{5}{8}$.)

I recently observed my six-year old preparing to play a game of war with his younger brother. He was removing all threes from the deck of cards. When I asked him why, his reply was something to the effect that since the threes couldn't take any other cards, they weren't needed. He was getting rid of them because they slowed down the game. Seizing the opportunity I dropped my newspaper and charged in with my best Socratic method forward. We discussed the war game that would result from the "new" deck of cards. In this game the fours would not be able to take any cards. Following his previous logic, we decided to remove the fours from the deck. Subsequent discussion led, in turn, to the removal of all fives, sixes and sevens. It was during the discussion of the eights that he showed the spark of insight that makes teachers (and fathers) know that their effort is all worthwhile. "But, Dad, pretty soon we'll only have aces and deuces and then the aces won't be any good!"

Following a discussion of the importance of the threes, I returned to my newspaper. My true effectiveness as a teacher became apparent to me when I later observed my sons playing the game—with one new rule. Threes take everything!

There is a very clear, three-part moral to this story:

Everyone knows that the big fish eat the little fish.

Daddies see this as fair and recognize that this makes the little fish very important.

Little boys see the inequity in this and promote the little fish to whales.

NEWS RELEASE FROM SSMA

Career education and its implications for science and mathematics teachers are the theme of the special issue just published by the School Science and Mathematics Association. The recent emphasis in the schools on career education has stressed that teachers from elementary school through the university level are to have extensive input into career education of their students. What input have mathematics and science teachers had in the implementation of career education in their schools? Will career education be a vehicle through which more students become interested in mathematics and science? The articles in this special issue provide some of the answers to these questions.

Some of the topic areas of individual articles include the history of career education and the legislative and funding programs during the last five years; how science and mathematics can be infused with career education in the elementary schools; introducing career education into secondary school science and mathematics classes; how career education can play a larger role in the junior colleges; how teacher education can facilitate including science and mathematics in career education and a comprehensive listing of resources for getting more ideas for infusing career education in science and mathematics classes.

Career education is interesting and challenging. Today, more than at any other time in our history, mathematics and science teachers should be aware of how their disciplines can become the first stepping stone to a life-long career. Single copies of this publication are available for $1.50 from the School Science and Mathematics Association, P. O. Box 1614, Indiana University of Pennsylvania, Indiana, PA 15701.

Minicalculators in our Schools 1975

By Joseph R. Caravella, NCTM

We have just entered the AGE OF THE MINICALCULATOR. Pocket calculators are now appearing in our society with a frequency approaching that of the pocket transistor radio. The price of the basic minicalculator has dropped below $10. And, the National Council of Teachers of Mathematics (NCTM) continues to endorse the minicalculator as a valuable instructional aid for mathematics education and to recommend the use of the minicalculator in the classroom.

> "With the decrease in cost of the minicalculator, its accessibility to students at all levels is increasing rapidly. Mathematics teachers should recognize the potential contribution of this calculator as a valuable instructional aid. In the classroom, the minicalculator should be used in imaginative ways to reinforce learning and to motivate the learner as he becomes proficient in mathematics."

The position statement above, adopted by the NCTM Board of Directors in September, 1974, is still relevant today. At its September 1975 meeting, the NCTM Board of Directors approved a report from the Council's Instructional Affairs Committee that identified nine ways in which the minicalculator can be used in the classroom:

1. To encourage students to be inquisitive and creative as they experiment with mathematical ideas
2. To assist the individual to become a wiser consumer
3. To reinforce the learning of the basic number facts and properties in addition, subtraction, multiplication, and division
4. To develop the understanding of computational algorithms by repeated operations
5. To serve as a flexible "answer key" to verify the results of computation
6. To promote student independence in problem solving
7. To solve problems that previously have been too time-consuming or impractical to be done with paper and pencil
8. To formulate generalizations from patterns of numbers that are displayed
9. To decrease the time needed to solve difficult computations.

In an article appearing in the current issue of TODAY'S EDUCATION, published by the National Education Association, entitled "A Calculator in Their Hands . . . The Minicalculator in Our Schools," Dr. E. Glenadine Gibb, the president of the NCTM, states that

> *"Creative use of minicalculators after the mathematical understandings have been extracted will establish the minicalculator as a valuable asset among the collections of instructional devices already found in today's mathematics classroom."*

The NCTM, through its Instructional Affairs Committee, its conventions, its affiliated groups, and its official journals, the MATHEMATICS TEACHER and the ARITHMETIC TEACHER, will continue to identify and share imaginative ways of working with minicalculators in the mathematics classroom.

Key to Algebra, Integers: Booklets 1-4 (Tj, Ts, S), Peter Rasmussen. Key Curriculum Project, Box 2304, Station A, Berkeley, CA 94702, 1971, 1972. 31 pp. each

This is a great set of small pamphlets that provide practice in the basic algebraic skills. Although they are called "text-workbooks" and could be used to individualize algebra instruction for general math students, the uses are many and varied. A year ago I purchased a set of *Booklet 3* to review skills in a basic algebra class—it was the best ten dollars I ever spent! The students enjoyed them, and this teacher ended the year with the satisfaction of having the best set of finals I've ever checked. Each booklet beyond the first begins with a brief review of concepts covered in prior booklets. The presentation is in very legible script rather than printing; the examples are explicit and easy for students to follow even without teacher aid, and the practice is sufficient to master the skill. Briefly, *Booklet 1* deals with prime numbers, addition, subtraction, and multiplication of integers; *Booklet 2* contains order, phrases, exponents, like and unlike terms; *Booklet 3* has evaluating phrases and solving equations; *Booklet 4* deals with polynomials, the distributive principle, and multiplying and factoring polynomials. Do take a look at these booklets and see how many ways they can relieve your problems with your less able students. —Munro.

Keystone Student in Olympiad

Bernard Beard from Keystone in San Antonio was one of the eight students representing the USA in the Fourth Olympiad held on May 6, 1975. The five top-scoring nations were, in the order of standing, Hungary, East Germany, United States, The USSR, and Great Britain. There were about twenty nations competing in the Olympiad. Congratulations to Bernard, to his teachers, and to his school.

REPRINTED FROM—

1—The Oklahoma Council of Teachers of Mathematics NEWSLETTER, Spring, 1974

2—Minnesota Council of Teachers of Mathematics NEWSLETTER, May, 1976

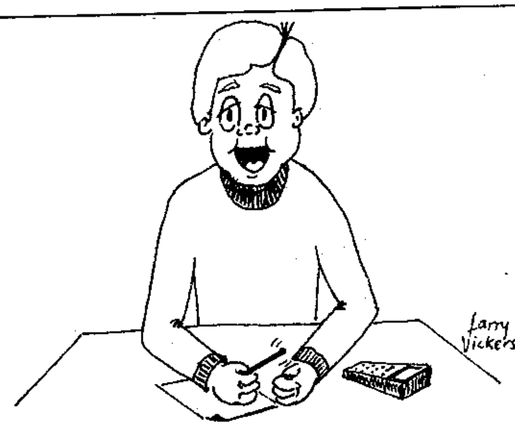
3—The Oklahoma Council of Teachers of Mathematics NEWSLETTER, Fall, 1974

4—Los Angeles City Teachers' Mathematics Association, THE CALCULATOR, December, 1972

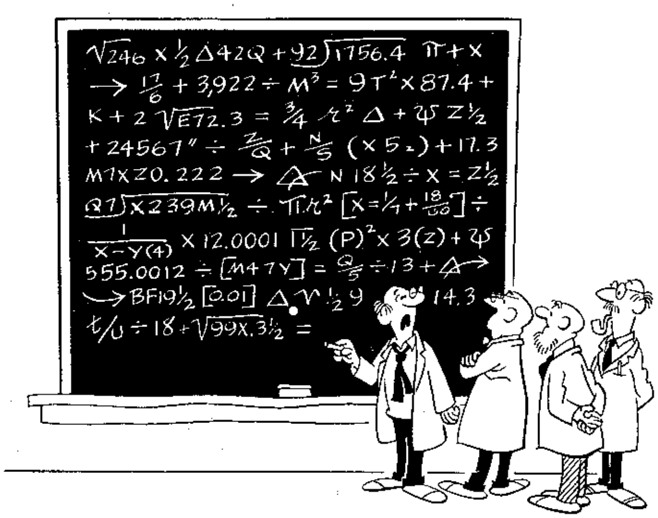
"I HOPE I'M NOT GOING TOO FAST FOR YOU!" 1

$$\begin{pmatrix} H & I & P \\ H & I & P \\ H & I & P \end{pmatrix}$$

This is known as the hip - hip - array. 2



"I COULDN'T DO MY ARITHMETIC LAST NIGHT, TEACHER . . . THE BATTERIES IN MY HAND CALCULATOR WERE DEAD!" 3



"Of course, if the dealer blackjacks, forget the whole thing." 4

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	$11.00 dues and one journal ☐ *Arithmetic Teacher* or ☐ *Mathematics Teacher*.	
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	6.00 additional for subscription to *Journal for Research in Mathematics Education* (NCTM members only)	
	.60 additional for individual subscription to *Mathematics Student Journal* (NCTM members only)	
	The membership dues payment includes $4.00 for a subscription to either the *Mathematics Teacher* or the *Arithmetic Teacher* and 25¢ for a subscription to the *Newsletter*. Life membership and institutional subscription information available on request from the Washington office.	

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