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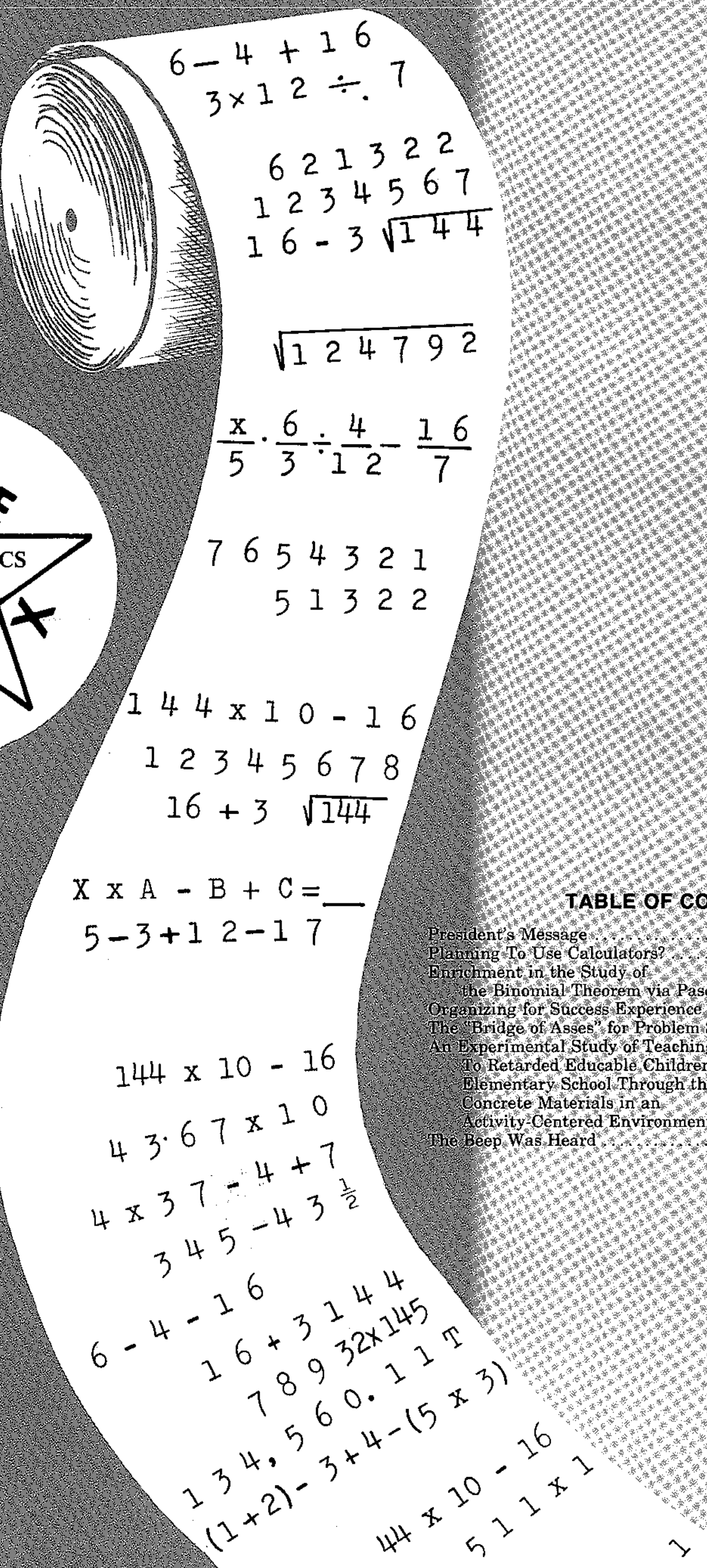

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President's Message

As we return for another year of school my hope is each of you had an enjoyable and profitable summer. With the new year upon us the opportunity for service will be springing up. Perhaps this can be our finest year ever.

In order to make this an outstanding year, may I suggest the following:

1. Be involved. Be a part of all that is taking place. Surely as mathematics teachers you will want to become involved with local, state, and national mathematics organizations as well as other organizations.

2. Be alert. Be sensitive to the problems and needs of those, both students and fellow teachers, around you. I am convinced the happiest teachers are those who are caught up with the people around them, sharing in their experiences in a personal way.

3. Be active. So many times members of any organization talk about what "they" are doing. As our membership becomes active, vocal, and involved so does our organization become a viable one. Encourage your fellow teachers to join us. As our organization grows and reaches into each building another opportunity for better mathematics education for all children arises.

4. Be professional. As mathematics teachers it is important that we support our professional organizations at every level. Through this we can have a more vital and active input into the direction and implementation of mathematics in our educational system.

We hope to have a newsletter out in the early fall. If you have materials you wish considered please send them to me.

Please remember CAMT in Austin on December 4-6. Nancy Ogden has put together an excellent program. I hope you will encourage your district to allow you to attend.

If you are not already aware of it we have lost a great friend of mathematics education in the retirement of Irene St. Clair from TEA. I am certain that Irene will find many ways to share her wealth of experience and knowledge with us even though she has stepped into retirement.

Many thanks in abundance to the officers and all of you who have been so gracious to help through advice and suggestions this past year. I have made numerous errors and hopefully they will diminish this year. Our membership list has been updated and this has been a tremendous task. We will send the membership reminders again this year. This helps us keep addresses current. If you received a reminder and have already renewed your membership, please do not be disturbed. It is the best means we have found to be certain everyone has been contacted.

A good, happy, profitable, and meaningful school year to you all.

BILL ASHWORTH

CAMT CONFERENCE FOR THE ADVANCEMENT OF MATHEMATICS TEACHING

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Planning To Use Calculators?

CHARLES A. REEVES

6th-Grade Instructor
The Florida State University

The day of the electronic calculator has dawned! For some of us this device appears to be another "giant step for mankind." Others of us predict that the calculator will soon fall by the wayside as merely another gadget that, at best, adds nothing to the classroom teaching of arithmetic or mathematics. After actually using them in a classroom, though, most math teachers would place themselves somewhere between the two extremes in feeling that the electronic calculator proves beneficial to students if it is used prudently.

The thoughts expressed below come from the three years of planning, experimenting and groping associated with trying to modernize my own classrooms by adding and using a classroom set of calculators. First the objectives I try to achieve are given, then some helpful hints to someone about to embark on the same endeavor. The observations and suggestions are based on intuition and subjective conclusions rather than statistical data gathered from well-controlled experiments.

The calculator goals that I strive for in my sixth-grade classes are these:

(i) to give students a skill they can use later in life. Regardless of how *I* feel about calculators personally, they *will* be used by every literate person within twenty years. And part of the function of our public school system is to prepare students for adult life.

(ii) To provide immediate reinforcement when arithmetic computational skills are being practiced. The advantages of the device in this role appear obvious and overwhelming. The disadvantage of students becoming lazy and not doing the practice itself occurs infrequently.

(iii) to eliminate computational mistakes (thus reducing frustration) that plague some students when they attack word problems that they know how to solve.

(iv) to (occasionally) introduce new concepts. For example, operations with positive/negative numbers can be introduced by "seeing what answer the calculator gives," assuming it's correct, and discovering a process for getting the same answer. This does not replace a later explanation built on logic, of course. Other topics that lend themselves to this approach are percent and decimals.

(v) to make math class more palatable. A certain amount of variety is essential for students to enjoy any school subject, and arithmetic is certainly no exception. Using an electronic calculator part of the time helps break up the monotony.

(vi) to have each student realize *when* it's appropriate for him/her as an individual to use a calculator. Most students in the sixth grade soon give up doing problems like " 9 × 8 " on the calculator — it's too much trouble to get it out for such a simple problem. But there are certain types of problems for which *everyone* should use a calculator, if it's handy (like balancing a checkbook).

In developing and striving to reach the goals above, the author has stumbled over, around and through most of the practical and pedagogical pitfalls that entrap someone venturing into an unknown area. The questions posed below are faced by anyone who is considering the use of calculators in a classroom—the answers provided suit my own situation, but might prove beneficial to someone in similar circumstances.

Q. I have limited funds—should I purchase "off brands" for less, or stick with the more expensive major brands?

A. Stay with brand names from reputable vendors. You'll be glad you did when one goes on the blink, and believe me, one will!

Q. What type should I buy?

A. The simpler the machine, the better! Essential features are the four operations, fixed and floating decimal point, and an easy-to-read display. A percent key is nice—a memory unit is not essential but could be useful in individualizing problems for advanced students. Tape read-out models are presently too expensive for most schools to purchase. Desk models that plug in are sturdier and better, generally speaking—if you can put up with a network of extension cords.

Q. How many should I purchase?

A. No more than 3 students/calculator is a nice goal, possibly to be reached in two or three years or in collaboration with other teachers. A lower ratio doesn't seem all that advantageous, but a higher one contributes to an atmosphere of horseplay and dependence on group leaders. At 1975 prices, a classroom set of ten could be purchased for less than $400 (using the TI-3500 as an example).

Q. What else will I need in my room if I use calculators?

A. Storage space for the calculators is essential, preferably where they can be locked up when not in use. The temptation for students, other teachers, and custodians to play around with them is just too great! You'll also need extension cords. A non-essential item to consider is a kit (several calculator companies make them) containing problems especially suited for calculator solutions.

Q. How can I prepare myself best for using calculators in my classes?

A. Become familiar with the instruction manual. Before students start asking questions, you need to know such things as "for the problem 7 − 5, do I press '−' before or after the '5'?" Also, can the machine multiply and divide with negative numbers, and how large a numeral can be displayed?

Develop your own set of calculator goals (similar to the list on pages 1 and 2) and give alot of thought as to appropriate activities to reach those goals. Plan to use the calculators once or twice a week, and for a variety of types of tasks.

Q. How can I prepare the students for the advent of the calculators?

A. Let them know a few weeks in advance that they will be allowed to use the calculators soon. Discuss with them the goals that you have established and the ways that you plan to meet those goals.

Q. How can I get students to take care of the calculators?

A. The best way is to convince them early that this little machine can save *them* alot of time and trouble and grief. It can take over some of the laborious tasks they would otherwise have to perform by hand (like adding long columns, multiplying large numbers). Once they realize its value to *them*, they'll take care of the machine. This also eliminates the distasteful and unsuccessful "scare tactics" we are so tempted to employ.

Q. When I run out of ideas for using calculators in my room, where should I turn?

A. Be ever on the lookout for books and other publications that will keep you involved and charged-up. Attend professional meetings and workshops in your area that involve calculators in the classroom.

If you haven't given much thought to classroom use of calculators before reading this article, please do so. If you have considered it and are still "riding the fence," please believe me—integrating them into your regular classroom routine is well-worth the effort and financial investment, if it's handled carefully.

Enrichment in the Study of the Binomial Theorem via Pascal's Triangle

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When teaching the binomial theorem in high school, some interesting enrichment can come from exploring Pascal's triangle. In this article we will assume the truth of the binomial theorem, show its relationship to Pascal's triangle, and discuss a number of observations from the triangle that can be used to enrich and motivate the study of the binomial theorem.

A familiar form of the binomial theorem appears below, where n is a positive integer and a and b are variables representing any two nonzero numbers.

$$(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1 . 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 . 2 . 3}a^{n-3}b^3 + \dots + \frac{n(n-1)(n-2)\dots3.2.1}{1.2.3\dots(n-2)(n-1)n}a^0b^n.$$

When n is given values in the order listed in the set [0,1,2,3,4,5 . . .], the binomial theorem gives us the following binomial expansions.

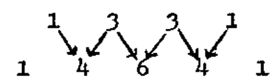
$$(a + b)^0 = \longrightarrow 1$$
$$(a + b)^1 = \longrightarrow 1a + 1b$$
$$(a + b)^2 = \longrightarrow 1a^2 + 2ab + 1b^2$$
$$(a + b)^3 = \longrightarrow 1a^3 + 3a^2b + 3ab^2 + 1b^3$$
$$(a + b)^4 = \longrightarrow 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$
$$(a + b)^5 = \to 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

If we write only the numerical coefficients (coefficients) of these binomial expansions in the same order and form as above, we have Pascal's triangle.

Row Number	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1
9	1 9 36 84 126 126 84 36 9 1

This triangular array of numbers can be traced back to the twelfth century mathematician Omar Khayyám, who is known to have discovered the binomial theorem. It later acquired the name of the French mathematician, Blaise Pascal, as a result of his work, "On the Arithmetic Triangle", which he wrote in 1654. Pascal related the triangular array to the binomial theorem and gave many other discoveries concerning the array.

Each term of Pascal's triangle except the "1's", can also be generated by finding the sum of the two terms directly above and to the left and right of it, as demonstrated below in finding the fourth row from the third.



With the coefficients of the binomial expansions in the triangular array form, some interesting observations can be made.

Observation 1. We can think of finding the coefficients of the expansion of $(a + b)^n$, as a process of finding combinations of n things, taken k at a time.

For example, to find the expansion of $(a + b)^3$, we write

$$(a + b)^3 = (a + b)(a + b)(a + b),$$

and successively select one variable from each of the three factors to obtain the sum of eight products

$$aaa + aab + aba + baa + bba + bab + abb + bbb.$$

Then combining similar terms and writing in exponential form, we have

$$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3.$$

We observe that from selecting one variable from each of the three factors of $(a + b)^3$, there is (are)

1 way of selecting 3a's and 0b's,
3 ways of selecting 1b,
3 ways of selecting 2b's
1 way of selecting 3b's.

We see then that these coefficients represent the number of ways in which 0, 1, 2, or 3 elements can be selected from a 3-element set. In general, it happens that the coefficients of the binomial theorem are combinations of n things taken k at a time, which is often denoted by $\binom{n}{k}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, and where $\binom{n}{0}$ is defined to be 1.

(See reference (2).

Pascal's triangle then may be rewritten using combination notation.

$$\begin{array}{c} \binom{0}{0} \\ \binom{1}{0} \quad \binom{1}{1} \\ \binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\ \binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \binom{n}{3} \quad \cdots \quad \binom{n}{n-1} \quad \binom{n}{n} \end{array}$$

The answer to a problem such as the following can be read directly from Pascal's triangle. If a president of a class selects a committee of three persons from a group of nine, how many different committees can he or she select? To answer this question, the person needs only to find the row where $n = 9$ in the triangle and count four entries from the left to obtain 84, the number equivalent to $\binom{9}{3}$

Observation 2. Since the coefficients in the binomial theorem are combinations, the binomial theorem can be rewritten as

$$(a + b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0 b^n,$$

where n is a positive integer. Using summation notation, the binomial theorem may be written in the compact form

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Observation 3. The coefficients of each binomial expansion is a power of 11.

$$\begin{array}{ccccccccc|l} & & & & 1 & & & & & = 11^0 \\ & & & 1 & & 1 & & & & = 11^1 \\ & & 1 & & 2 & & 1 & & & = 11^2 \\ & 1 & & 3 & & 3 & & 1 & & = 11^3 \\ 1 & & 4 & & 6 & & 4 & & 1 & = 11^4 \end{array}$$

This generalization requires that we not "carry digits" in the addition involved in the multiplication algorithm. For example,

$$11^5 = 11^4 \cdot 11 = 14641 \cdot 11 = \begin{array}{cccccc} & 1 & 4 & 6 & 4 & 1 \\ \times & & & & 1 & 1 \\ \hline & 1 & 4 & 6 & 4 & 1 \\ 1 & 4 & 6 & 4 & 1 & \\ \hline 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

Notice that we have not used the "carrying" process. Instead, we have written the column sum of 10 in each case, keeping in mind that the 10 farthest to the right denotes 10 hundreds, while the other 10 denotes 10 thousands. (See reference (3)

Observation 4. The sum of the coefficients of the expansion of $(a + b)^n$ is 2^n.

$$\begin{array}{ll} n = 0 & 1 = \longrightarrow 2^0 \\ n = 1 & 1 + 1 = \longrightarrow 2^1 \\ n = 2 & 1 + 2 + 1 = \longrightarrow 2^2 \\ n = 3 & 1 + 3 + 3 + 1 = \longrightarrow 2^3 \end{array}$$

This observation can be stated in compact form as

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

and can be proved by using mathematical induction.

Observation 5. The partial sum of the entries of each diagonal can be found in the next row and slightly to the southeast.

For example, the third diagonal, which has the entries 1, 3, 6, 10, 15, . . . , has partial sums as indicated below.

$$\begin{array}{l} 1 = \longrightarrow 1 \\ 1 + 3 = \longrightarrow 4 \\ 1 + 3 + 6 = \longrightarrow 10 \end{array}$$

Find in Pascal's triangle, the entries corresponding to the partial sums: a) $1 + 3 + 6 + 10$ b) $1 + 3 + 6 + 10 + 15$.

Observation 6. The number of lines determined by n ($n > 1$) distinct points in a plane with no three points collinear, is the third coefficient in the expansion of $(a + b)^n$, or $\binom{n}{2}$. Thus, it is an entry in the third diagonal of Pascal's triangle.

Number of Points (n) (Number (n) of row)	Number of lines $\binom{n}{2}$
2	$\binom{2}{2} = 1$
3	$\binom{3}{2} = 3$
4	$\binom{4}{2} = 6$

Observation 7. The sum of the squares of the coefficients of the expansion of $(a + b)^n$, can be found in Pascal's triangle, n rows below the row corresponding to these coefficients, or in other words, the row with number $2n$.

For example, the sum of the squares of the coefficients of $(a + b)^3$ is the sum of the squares of the third row, that is, $1^2 + 3^2 + 3^2 + 1^2$ or 20.

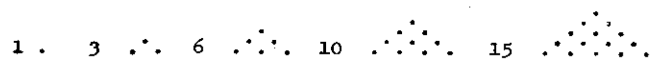
1
1 1
1 2 1
1 3 3 1 3rd row
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1 6th row

Then as expected, the 20 is found in the center of the sixth row. Since 20 can also be written as $\binom{6}{3}$, we may write $\sum_{k=0}^{3} \binom{3}{k}^2 = \binom{2 \cdot 3}{3}$.

It happens that in general, $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

(For a proof of this generalization, see reference (5).

Observation 8. The entries in the third diagonal of Pascal's triangle name triangular numbers, that is, numbers that correspond to the same quantity of dots arranged in a triangular pattern.

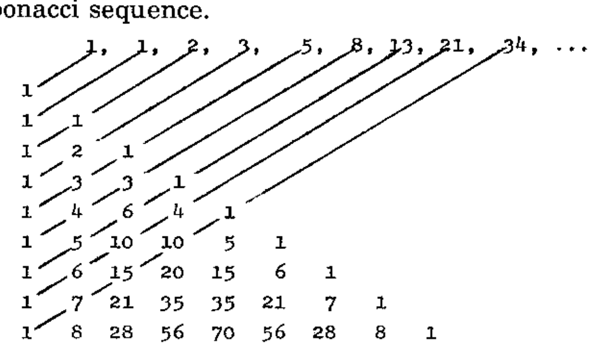


Observation 9. The number of diagonals of a polygon with n sides is a number in the second diagonal of Pascal's triangle named by the expression $\frac{n^2 - 3n}{2}$.

Number (n) of Sides of Polygon	Number of Diagonals
3	0
4	2
5	5
6	9
⋮	⋮

Describe another way of determining the number of diagonals of a polygon from Pascal's triangle. Do you observe a pattern for these numbers in the second diagonal?

Observation 10. If Pascal's triangle is in a slightly different form, the sums of the diagonals is the Fibonacci sequence.



Observation 11. Probabilities of certain events occurring or not occurring can easily be determined from the coefficients of the expansion of $(a + b)^n$.

For example, to find the probability of tossing two heads and one tail in a toss of three coins, we need only take the sum of the entries in the third row of Pascal's triangle as the denominator of the probability ratio and take for the numerator the entry in the third row that corresponds to the term containing the expression h^2t^1. Thus, the desired probability is ⅜. What is the probability of tossing three heads in one toss of three coins? (To determine why this procedure works, see reference (1) or (2).)

Using Pascal's triangle, the teacher can effectively enrich and motivate the study of the binomial theorem. A teacher can lead a high school student to discover several of the "Observations" discussed above. Some of this enrichment material might be used by a teacher without requiring additional class periods of time, but if all that is discussed here is used, more time would be required. Also, this material might be used for student presented programs in a mathematics club, where more discovery time could be made available. (See reference (4).

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Organizing for Success Experience in Mathematics

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How is it that we have been able to take something as close to our lives as mathematics and make the study of it so dull, so solemn, so dry, and so unattractive?! Perhaps no other area of the elementary school curriculum is approached in such a dour manner.

A major reason for this is that so many people don't believe in the possibility of success for all children. Teachers, and parents too, often describe their own dislike of mathematics without shame and without concern. Mathematics is simply not a likeable subject! Or is it?

Kids have every right to expect that mathematics, like every other school subject, can be enjoyable. We can provide this enjoyment if we organize classroom instruction to provide a continuing series of success-oriented experiences. To do this, we need to identify and use behavioral objectives in our instruction to better advantage than we have in the past. Carlton Washburne wrote more than fifty years ago, "A clearly defined and desirable goal leads one to accomplishment almost regardless of method."[1] From the time of this statement to present concerns for accountablity, the role of objectives in mathematics instruction has occupied much of our attention. In fact, so much attention has been devoted to determining what the goals should be that we have given too little attention to how these goals should be used in instruction. Creative use of goal statements can help us get the right mathematics, the right experience, the right practice—all at the right time for the right child. When all this is right, children do experience success.

[1]Washburne, C. W. "Motives and Goals in Education," *Journal of Education:* 96: 92, August 10, 1922.

Goals must be clearly shared with learners for knowledge of one's achievement in mathematics can do much to create positive attitudes toward mathematics. Although we do not know how pride in one's achievement begets positive attitudes or how positive attitudes beget improved achievement, the relationship between achievement and attitudes deserves more attention than it has been given. Success does generate success. Providing success experiences for every child, though difficult, is possible for each teacher. The task is not simple, but it can be accomplished through adherence to a series of steps, and the following suggestions are made to provide a starting point for teachers who want to begin thinking and practicing a success mode of instruction.

1. *Believe that mathematics instruction can and should be organized so all children in our classes experience success in mathematics.*

It all begins here. One must believe in success before one can bring it about in the classroom. Surprisingly, for many teachers this first step will require considerable thought and self-analysis. It takes more than simply saying, "Yes, that is true. Children should succeed in mathematics." It takes a belief strong enough to result in the commitment to begin making it possible for all children in your classroom to succeed.

2. *Organize instruction around major topics for concentrated periods of study.*

The lesson sequence that too frequently changes direction or that includes many emphases running concurrently results in a very limited and obstructed view of goals for instruction. A child sees the direction for study as simply turning to the next page in the book. Learning experiences organized around major topics allow the child to understand a specific goal and to work with that until it is fulfilled. Children whose study topic changes every day or two lose the sense of direction their learning is taking.

3. *Focus on relatively few (20 to 30) major objectives for a given year of study in mathematics.*

Twenty to thirty major objectives make it possible for both teachers and students to comprehend the totality of the year's work. Many teachers have been involved in the task of writing dozens, if not hundreds, of objectives for a given year. This may serve the purposes of those responsible for curriculum development, but for the purposes of instruction there should be only as many year-end objectives as a teacher can keep constantly in mind.

4. *Provide each child with a list of the major objectives stated in terms that have meaning to the child.*

Keeping a list of all objectives and checking off those that have been mastered provide many benefits, the most important being the child's sense of success as objectives are checked off. This implies that in many classrooms, objectives for the year may be differentiated appropriately for children of different abilities.

5. *A substantial amount of sharing and planning time should be devoted early in the year to the establishment and understanding of the objectives.*

It is not only important that a child has a list of the objectives for the year, but also that their intent is well understood. Providing children with such a map of the year's work does much to motivate learners. Start with a master list of objectives and a sense of the manner in which the class is to be organized. Then determine the appropriateness of this set of objectives for all children and make the adaptations needed for groups within the class, or for individual learners.

6. *Make sure the child understands when a topic is due for mastery.*

Children become so accustomed to being admonished to "memorize" or "master" that they think of each topic in mathematics as something to be mastered when first presented. This certainly is not intended and is, of course, unrealistic. Readiness, introduction, development, and then mastery should be the sequence of instruction. Letting the child in on it, letting the child know that ideas will first be explored, letting the child know that it is only later that the same topic which is now being explored will be practices to mastery: these things are important in sequencing instruction.

7. *Celebrate success when it happens!*

How nice it is to celebrate success! How seldom it is done! To finally have mastered the multiplication facts is a major accomplishment; to be able to rename almost any fraction is quite a feat; to recognize geometric shapes can be a joy. Quietly in most instances, and with greater fanfare in others, with signs, bulletin board announcements, a note home to parents, or a verbal pat-on-back, teacher and child share a celebration. Success is when the child says, "I can."

The "Bridge of Asses" for Problem Solvers

— **BARNABAS B. HUGHES, O.F.M.**

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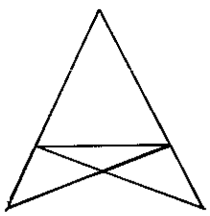
During the early Middle Ages the study of plane geometry in European schools rarely proceeded beyond the first book of Euclid's *Elements*. Many students, in fact, got "hung up" on and stopped at the proof of the fifth proposition, *The base angles of an isosceles triangle are equal; and if the equal lines are* *produced farther, the angles under them are equal.* The diagram used in the proof is



The theorem came to be called *pons assinorum,* or "Bridge of Asses,"[1] because an ass could not cross it. Quite literally, the ass would be hung up on the top point were it to climb so far.

The "Bridge of Asses" in problem solving is the distinction between, and separation of, data and condition in a word problem. The inability to recognize these leaves the student incapable of writing the necessary equation(s) or constructing the geometric figure that yields the solution. Such a deficiency breeds frustration which, in turn, begets the negative attitude characterized by the remark voiced too often, "Mathematics was my worst subject." One legitimately wonders if this inability stems from a lack of discriminating instruction. If the difference between data and condition are not clear to the teacher, if concepts and examples are not clear to students, few students will be able to discriminate between the two, much less recognize each.

Training in data/condition recognition is done best in its proper context: understand the problem. Thus, to those two is added recognition of the unknown. The teacher puts his students at an advantage who first has them analyze all the problem assigned for homework in order to identify the three essential elements—unknown, data and contion. In this kind of study, a classroom oral exercise or a small group activity, the word problems are only taken apart; they are not solved, for solution is not the object of the exercise. Indeed, if a student cannot separate the components of the problem, he cannot solve it.

The first component, *recognizing the unknown,* is generally the easiest to master. The unknown may be the simple predicate of a question—what are the *ages* of the children, or of an imperative sentence—find the *number* of cattle. Whatever is sought is the unknown. In geometric problems-to-find,[2] the unknown will be a point, line, figure of measure; for instance, draw the triangle, construct the parallel lines, measure the angle.

Ordinarily, word problems state something as given. This is the second component, the *data:* given *three* noncollinear *points* in position, draw a triangle; divide *$ 90* exactly between three favorite nieces; find four odd integers *whose sum is 14*. The data ties the solution to unalterable specifics. Change the number of points or the amount of money or the sum, and the entire problem becomes a different one. Whatever the data, at the beginning of serious study of word problems, it should be easily recognizable; that is, the data should be explicit. (The data may be *implicit* in the condition, a phenomenon to attend to after discussing the meaning of "condition.")

The third component, the *condition,* expresses a relationship between the data and the unknown, or between the unknowns. It changes neither the data nor the unknown; rather, it is the way the problem looks at them. In the above example, the three *non-collinear* points are given *in position.* The condition is that the points are noncollinear; this is the essential relationship between the three. A secondary condition is *in position,* which can be changed and the triangle still drawn. Make the points collinear, however, and the students will balk at accepting a straight linesegment as a triangle (although this maneuver may lead them to new concepts). Whatever the change in the relationship, there is no altering the data—the three points, nor the unknown—a triangle.

Again: the $ 90 is to be divided *equally*. The adverb expresses a relationship between the parts of the desired unknowns. While *equally* affects the partition of the data, it does not effect any change in the quantity of the data nor in the number of partitions. A change in the relationship among the desired parts to a ration of 1:2:3 does not alter the gross amount of $ 90; only the distribution of the money is affected.

Again: the desired numbers are *odd* integers. The adjective characterizes the kind of solution acceptable for the unknowns; all the unknowns must be odd numbers. Changing this relationship to finding even numbers does not affect the data—their sum. Nor would an alteration of the relationship of each number to itself affect the data, such as finding four rational numbers. The given sum remains constant as does the quantity of numbers to be found.

Occasionally, as remarked above, the data may be *implicit* in the condition. This makes the separation of data from condition more difficult. Consider this example:

> In eight years a boy will be three times as old as he was eight years ago. How old is he now?

The *unknown* is clear, the present age of the boy. But the *data* and the *condition* are worded closely together; namely, the two "eight years" and the "three times." Which is data? which is condition? *The element most closely connect to the unknown is the data.* Clearly, the unknown is coupled with the pair of "eight years," thus:

$$x - 8 \quad \text{and} \quad x + 8.$$

These two expressions are related by the "three times." This factor creates the *relation of equality* between the two; and the equation follows easily,

$$x + 8 = 3(x - 8).$$

A change in the relating factor to two times does not affect either the eight years or the coupling. The fine line separating data from condition in this and similar problems permits an alternate description: given two elements (here, the eight years and the three times), the element which works on the other is the condition. Clearly, the "three times" affects the "eight years."

The effect of the condition "working on the other" is one or more equations. Newton puts the matter succinctly, "And any question being proposed, Skill is particularly required to denote all its conditions by so many equations."[3] In other words, for each condition there is an equation. Herein lies an additional aid for distinguishing data from condition—*the equation represents the condition.* Hence, the components of the equation are the unknown and the data, suitably related. An example clarifies this point.

An important problem in Polya's *Mathematical Discovery* (important because of the significance of its general solution) is:

> A man walked five hours, first along a level road, then up a hill, then he turned around and walked back to his starting point along the same route. He walked 4 mph on the level, 3 mph uphill, and 6 mph downhill. Find the distance walked.[4]

The unknown is obvious—the total distance walked. Both time and rates are given. How to distinguish data from condition? For, it seems reasonable to say that a change in rates would effect a change in total time; hence, "rates" may stand out as the condition. On the other hand, a change in time would effect a change in the unknown; hence, "five hours" may appear as a likely candidate for the role of condition. To confound matters: if we use the principle *The element more closely connected to the unknown is the data,* it seems obvious that both rate and time are connected equally close to distance.

We slip between the horns of the dilemma *by looking at the information in the most natural way.* "Natural" means "easier." It is easier to relate the various rates to the different distances, than to relate the one time of five hours either to the unknown distance or to the stated rates. That is, algebraically, where x is the total distance walked and y the distance on the hill:

time on level:	time up:	time down:	time on level:	total time:
$\frac{\frac{1}{2}x - y}{4}$ +	$\frac{y}{3}$ +	$\frac{y}{6}$ +	$\frac{\frac{1}{2}x - y}{4}$ =	5 .

$$\frac{\frac{1}{2}x - y}{4} + \frac{y}{3} + \frac{y}{6} + \frac{\frac{1}{2}x - y}{4} = 5.$$

The equation says that the whole time of five hours consists of four part-times. Hence, in this problem, the condition is time.

The most natural ways of looking at the information in the problem separates condition from data. Discriminating these two bits of information is the apex of the "pons assinorum" for problem solvers.

References

[1]The translation "Bridge of Fools" has a different rational. The fool is weak in the ability to understand. Hence, the figure of speech could be applied to *any* proposition difficult to understand.

[2]For a clear explanation of the distinction between problems-to-*find* and problems-to-*solve,* see George Polya's *How To Solve It* (Princeton: Princeton University Press, 1971—2nd edition), pp. 154-57.

[3]Issac Newton, *Universal Arithmetick: or, A Treatise of Arithmetical Composition and Resolution* (1728), p. 67.

[4]*op.cit.,* I, p. 41.

An Experimental Study of Teaching Mathematics To Retarded Educable Children in Elementary School Through the Use of Concrete Materials in an Activity-Centered Environment

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In the summer of 1972, The Office of Mathematics Education of the School District of Philadelphia initiated a program of teaching mathematics to elementary school retarded educable children using concrete materials in an activity-centered environment. Teachers were provided training in the use of this approach during a two week summer staff development program. During the academic year, mathematics laboratories were set up in each of the participating teacher's classrooms.

A review of the literature has revealed a paucity of information on the use of concrete materials in the teaching of mathematics to retarded educable children. The purpose of this study was to determine the effectiveness of using concrete materials on the academic achievement in mathematics of elementary school retarded educable children as an alternative educational strategy to teaching mathematics in a classroom in which there is an absence of concrete materials.

It was hypothesized that six to nine year old and nine to twelve year retarded educable children in elementary schools experiencing experimental conditions would show greater growth in mathematics achievement than the control group in one school year. It was further hypothesized that teachers of retarded educable children in elementary schools experiencing experimental conditions would show a more positive attitude toward the teaching of mathematics in one school year.

Procedure

Forty-two School District of Philadelphia Public Schools were used for this study; twenty-one control schools were selected to match twenty-one experimental schools in pupil population, socioeconomic status, and geographic locations. Control and experimental groups were equated on chronological age and I.Q. The populations of both the experimental group and the control group were further subdivided into two age groups consisting of six to nine year old children and nine to twelve year old children. Analysis of variance was used to determine whether experimental and control groups were equated prior to the experiment. There were no significant differences between the groups on either chronological age or I.Q.

The six to nine year old experimental and control group children were administered the Individual Arithmetic Test for Educable Mentally Retarded Children Ages Six through Nine. The nine to twelve year old experimental and control group children were administered the Individual Arithmetic Achievement Test for Retarded Children. These tests were administered in September, 1972, and again in June, 1973. The YAT Attitude Toward Mathematics Survey was administered to the twenty-one teacher participants in August, 1972, and in June, 1973.

Results

Repeated measures designs were used in the analysis of data. These were used to examine pre-test and post-test scores in order to analyze gains made in growth. Analysis of variance and analnsis of variance for simhle effects was applied to total test scores, applications, operations, and contents for both the experimental and control groups of six to nine year olds and nine to twelve year olds. Analysis of variance was applied to the pre-test and post-test scores of the experimental group of teachers. No significant difference was found in the six to nine year old groups with regard to growth in mathematics achievement. The experimental group of nine to twelve year old children obtained significantly better gains in growth in mathematics achievement than did the control group of nine to twelve year old children. No significant difference was found in the results of the pre-test and post-test scores of the teacher participants in the project.

Conclusion

The findings indicate that the mathematics achievement of elementary school retarded educable children in classrooms involved in activities using concrete materials is the same as or better than the achievement of retarded educable children in classrooms not using this approach.

The Beep Was Heard . . .

by Ms. Evelyn Ussery

Blessing, Texas

The beep was heard around the world,
How loud the hue and cry,
Those Russians (ignorant though they be)
Had Sputnik flying high!

A nation moved to second place,
The pill hung in the craw,
Those Russians (peasants though they be)
Had beat us to the draw!

No longer could we trip at ease
Along the rose-strewn path,
Great minds combined and ushered in
The era of 'new math'!

Perhaps discussions went like this -
What age can comprehend
That some things can be infinite,
Instead of without end?

And would the young ones like to know
Of early, early times,
Erastothenes and how his sieve
Could find the numbers prime?

Collections now became a set,
The blank became a frame,
The x and y were variables,
All had the proper name.

And so the world of math was changed,
Concepts were emphasized,
The how-to-do-it was replaced
With understanding why.

A decade now has come and gone,
How loud the hue and cry,
It seems the Johnnys cannot add,
Or even multiply!

He may can draw the circles round,
From given radii,
But this is not so great a thing,
If he can't multiply!

Or work in bases two and five,
Is such as this a fad?
The most important thing by far,
Is Johnny learns to add!

Reform is in the wind again,
We stand upon the brink,
What think you should take precedent -
To multiply or think?

Can we return to programs old,
So pleasing to the eye,
Example A will tell you how,
And no one questioned why?

Encourage, yes, the skills of old,
To add and multiply,
But hope the Johnnys that you teach,
Will always question why.

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