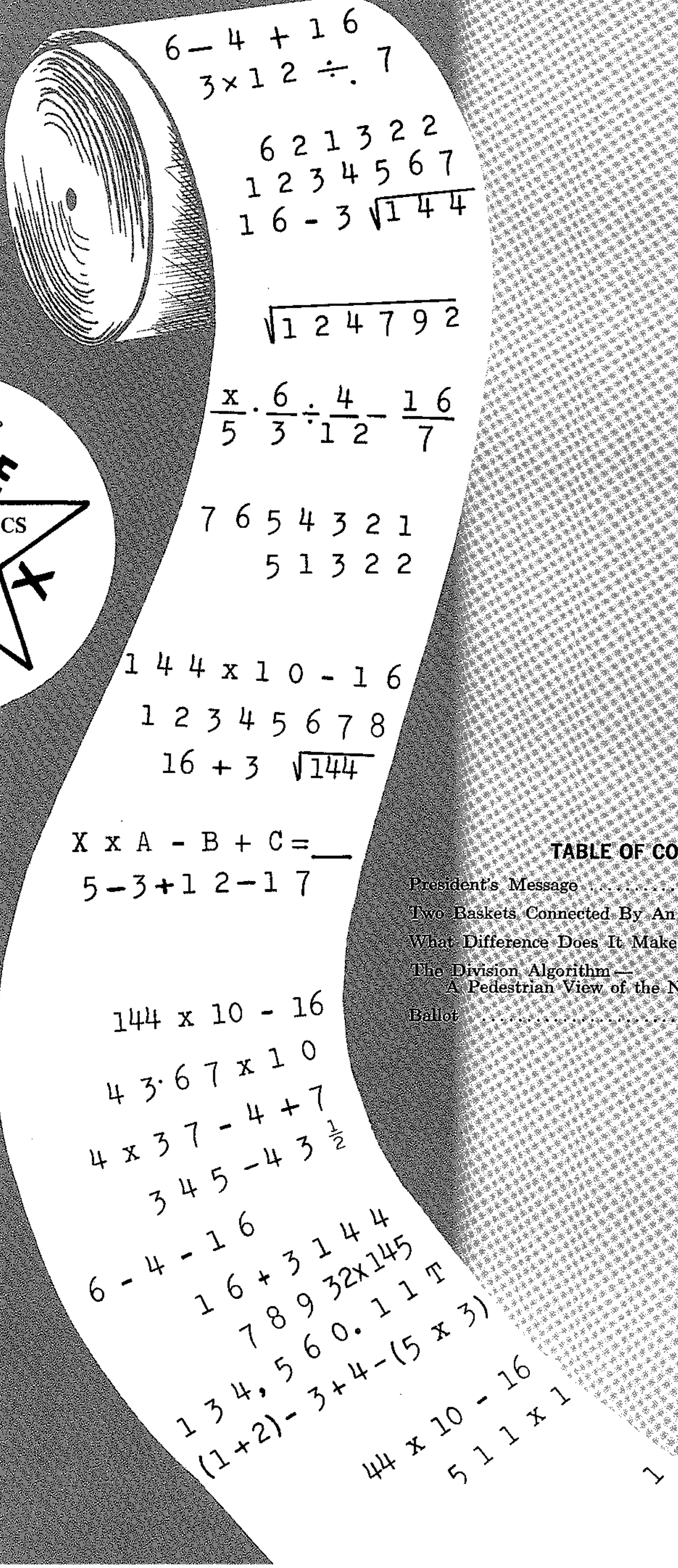
Vol. XXII No. 3

TEXAS MATHEMATICS TEACHER

May 1975



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TEXAS MATHEMATICS TEACHER is the official journal of the Texas Council of Teachers of Mathematics. The views expressed are the contributor's own and are not necessarily those of the publisher or the editor. All manuscripts and correspondence about this publication should be addressed to Mr. J. William Brown, *Texas Mathematics Teacher*, 100 So. Glasgow Drive, Dallas, Texas 75214.

TEXAS MATHEMATICS TEACHER is published quarterly by the Texas Council of Teachers of Mathematics. Payment of membership fee of $5.00 entitles members to all regular Council Publications.

President's Message

Another year has almost ended. With its events, both good and bad, we look back to reflect on "what has been" and are compelled to look forward to "what is to come?" On every hand we seem to be in an era of controversy. Perhaps this is good that we are questioning ourselves. It is a wholesome thing that we are questioning our curriculum in areas of content, depth, purpose, and approach. Perhaps changes are in order. If this is true, change for the sake of change will lead us nowhere, except possibly to the need for more change. It is my desire that when, and if, change is to occur that mathematics teachers will be positive in its introduction and implementation.

In this issue you have a ballot for election of officers for next year. It appears to be a very good selection you have for the two offices. Please return your ballot as quickly as possible—lest you forget and leave this undone over the summer.

By now you all have received a newsletter from T C T M. We need your comments and reactions to it. If we are to continue such a venture, it will be necessary to know your feelings concerning it.

Please note on your calendar and reserve September 13 for the T C T M workshop ot Dobie High School in Pasadena. The workshop leaders are being chosen now. You will receive a program around August 1. The leaders which we already have promise to make this one the best yet.

I wish for each of you the best for your summer vacation.

BILL ASHWORTH

Dates to Remember

AUGUST 20-22, 1975 : Honolulu NCTM Meeting

DECEMBER 4-6, 1975 : CAMT (TCTM) Austin, Texas

TWO BASKETS CONNECTED BY AN EQUATION

R. V. Andree
The University of Oklahoma
Norman, Oklahoma

Many interesting developments in mathematics have occurred during the last 200 years, yet most of what we teach is much older than that.

Euclidean Geometry comes to us from well over 2000 years ago. Furthermore the Babylonians knew much of geometry 1500 years before that.

Algebra seems a bit more modern, the name only dates back to the Arabic of about 825 A.D., but there is plenty of clear evidence in the Nippur (modern Nuffar) tablets that the Sumerians and the Babylonians could solve many linear, quadratic, cubic and even some quartic equations about 2100 B.C.

Analytic Geometry goes back to the time of Descartes (mid 1600s).

Calculus comes from the time of Newton and Liebnitz in the late 1600s, almost 300 years ago.

Yet the truth of the matter is that the majority of the current of mathematical knowledge has been discovered much more recently than this. Indeed, a recent study shows that more new mathematics was invented (or does one discover rather than invent mathematics?) in the fifty year period, 1900-1950, than in the 5000 year period 3100 B.C.-1900 A.D. Furthermore, it is observable that more mathematics was published in the 15 years 1950-1965 than in the highly prolific 1900-1950 period and that more mathematics was published in 1965-1972 than in 1950-1965. This implies that more new mathematics was discovered during the 22 years since 1950 than in all history before 1950. Let us at least tell our students about some of the mathematics of the last 200 years, if not the last two decades.

Much of mathematics has been discovered by *young* mathematicians. Let us look at one such mathematicians's contribution.

Evariste Galois (1811-1832) lead a brief, frustrating and tragic life. He was expelled from school and was imprisoned for his revolutionary activities. He twice failed his college entrance examinations. He was killed in a framed duel when he was 21 years old, but still he managed to found an entire branch of mathematics. Much of what Galois did is understandable to a good high school student providing modern notation is used.

You may examine the fruit of Galois' genius in your library if you would be interested in doing so.

Whether or not you do so, it will be important that your students realize *how the game of mathematics is played.* It is to this topic that we shall devote the rest of our attention.

Let us consider two baskets (sets, if you prefer) of *numbers* and an equation that connects them. The first basket will contain possible coefficients for our equation, while the second basket will contain all of the numbers which are solutions of the equation using any of the numbers from the first basket as coefficients.

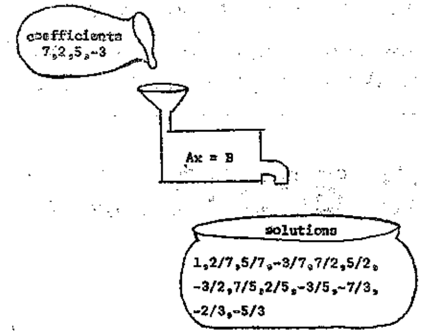
If the equation is $Ax = C$

and if the first basket contains only the numbers 7, 2, 5, and − 3, then the resulting equations are

7x = 7	2x = 7	5x = 7	−3x = 7
7x = 2	2x = 2	5x = 2	−3x = 2
7x = 5	2x = 5	5x = 5	−3x = 5
7x =−3	2x =−3	5x =−3	−3x =−3

and the solutions are

2/7, 5/7, −3/7, 7/2, 5/2, −3/2, 7/5, 2/5, −3/5, −7/3, −2/3, −5/3.



Thus, in this case, our solutions basket needs to be larger than the coefficient basket which contained only {7, 2, 5, − 3}.

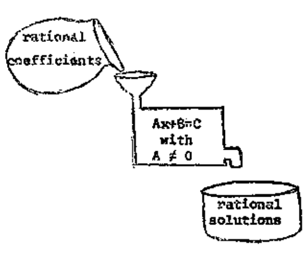
In general, we shall be particularly interested in baskets which contain infinite sets of numbers. For example, suppose the equations were again Ax = C and the coefficients A and C were permitted to be any *positive integers*. What can you say about the numbers which would be in the solution basket? Certainly every integer must be in the solution basket (1x = C) but so would a good many other numbers. If the solution basket contained all of the *rational* numbers, then every solution of Ax = C would be found in the solution basket—however, there would also be many rational numbers in the solution basket which would not be needed to solve Ax = C if A and C are *positive* integers. Can you suggest a smaller set of numbers for the solution basket?

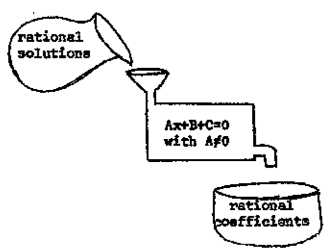
Let us assume for a moment that we do wish to permit the solution basket to contain all of the rational numbers. Can we now enlarge the permissible set of numbers in the coefficient basket without demanding additional numbers in the solution basket?

This, then, is typical of the game of mathematics. Let's examine our input and our output and see what happens to one when you change the other. Then try to make general statements that are valid for infinitely many cases. In our present discussion one possible final conclusion might well be, *"If A and C are integers with $A \neq 0$, then there is a rational number x which satisfies $Ax = C$"*; possibly we might go even further and say that *"Conversely if x is a rational number then there exists integers A and C with $A \neq 0$, such that $Ax = C$"*.

In discussing the solution of equations, it is vital to state from what set the coefficients are to be taken and in what set the solutions are to be found. If the coefficients A and B of $Ax + B = 0$ are to be the integers, then we *cannot* guarantee to be able to find an integral solution to $Ax + B = 0$ even if $A \neq 0$ since there is no integer x such that $3x + 5 = 0$. If the solutions are to be taken from the set of rational numbers, then every linear equation $Ax + B = C$ with $A \neq 0$ whose coefficients A, B and C are integers will have a solution. Indeed, we can enlarge the set from which the coefficients are chosen without being forced to further enlarge the set in which the solutions will be found since if

$Ax + B = C$, $A \neq 0$, with A, B, and C *rational* will always have a rational solution x.





Example:

$$5x + 2 = 9$$

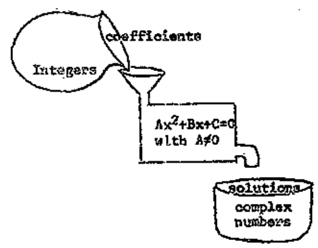
has a solution if x represents a number of dollars, but may have no solution if x represents a number of people since x = 1.40 dollars is meaningful, but 1.4 people is harder to visualize.

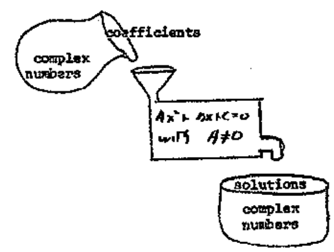
If we extend our consideration to quadratic equations we again discover if the coefficients A, B, C, with $A \neq 0$ are taken from the set of integers, that there exist equations such as $x^2 + 5x + 1 = 0$ or $x^2 + x + 3 = 0$ which have no integral solutions. Furthermore, even if the set from which solutions may be found is extended to the rational numbers, no solution will be found. If the set in which solutions may be hunted is extended to the real numbers, the equation $x^2 + 5x + 1 = 0$ does have a solution in the set of real numbers, but $x^2 + x + 3 = 0$ still does not. If we finally extend the set in which solutions may be hunted to the complex numbers then *every* quadratic equation $Ax^2 + Bx + C = 0$ with $A \neq 0$ and integral coefficients A, B and C will at last have a solution in the set of complex numbers.

Now that we have reached our current goal, we as mathematicians ask ourselves if we can enlarge the set from which the coefficients may be chosen without having to again enlarge the set containing the solutions. (That is, can we get more for our money?)

The answer is yes, yes, yes. Indeed, the set from which the coefficients can be chosen can be enlarged to the rationals, and can be enlarged to the reals and indeed *can be even enlarged to the complex numbers without having to further enlarge the set from which contains the solutions.* We have the theorem.

If A, B, C are complex numbers with $A \neq 0$, then the equation $Ax^2 + Bx + C = 0$ has a complex solution.



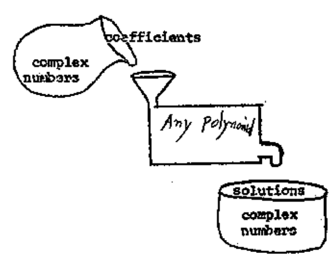


Now we are ready to make a tremendous generalization. Namely, let the equation be any *polynomial* equation

$$Ax^N + Bx^{N-1} + Cx^{N-2} + \ldots + Qx^2 + Rx + S = T$$

with $A \neq 0$,

then if the coefficients are taken from the set of real numbers, the solutions will be found in a basket of complex numbers. Furthermore, (although we shall not prove it here) if the coefficients are taken from the set of complex numbers, the solutions will still be found in the basket containing the complex numbers.



It is reasonable to consider other types of numbers such as the so called clock numbers (module 12 arithmetic). For what values A and C does the congruence

$$Ax \equiv C \quad \text{mod } 12$$

have a solution, and what are these solutions? What about

$$Ax \equiv C \quad \text{mod } 7$$

or

$$Ax^2 + Bx + C \equiv D \quad \text{mod } 13?$$

Such projects merit careful investigation and can produce worthwhile papers for publication in the *Math Log* or the *Student Journal* or for Science Fair projects.

Closely related to all of this is the question of whether or not it is possible to *factor* a given polynomial expression such as

$$x^2 - 9 \quad \text{or} \quad x^2 - 2 \quad \text{or } x^2 + 1.$$

The question has no meaning unless one also specifies the set from which the coefficients of the factors may be selected—in short the environment in which the factoring is to take place.

$x^2 - 9 = (x + 3)(x - 3)$ with factor coefficients from the set of integers.

But $x^2 - 3$ and $x^2 + 1$ are not factorable with integral coefficients.

The same statements hold if the coefficients of the factors are permitted to be rational numbers.

Both $x^2 - 9 = (x + 3)(x - 3)$ and $x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$ are factorable over the real number.

But $x^2 + 1$ is still not factorable with real coefficients.

However, if we permit our factors to have complex coefficients, then all three expressions are factorable.

$x^2 - 9 = (x + 3)(x - 3)$; $x^2 - 2 = (x \quad \sqrt{2})(x - \sqrt{2})$; $x^2 + 1 = (x + 1)(x - 1)$.

Hence, the environment in which one works may easily influence whether or not a given task is possible or impossible in mathematics as in real life.

A problem that puzzled mathematicians for many generations was whether or not it was possible to determine solutions of polynomial equations using only the exact coefficients of the polynomial and applying the operations of addition, subtraction, multiplication, division and the extraction of roots *a finite number of times*. Every algebra student learns that the equation

$$Ax^2 + Bx + C = 0 \text{ with } A \neq 0$$

has the solution

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

which are obtained by operating a finite number of times on the original coefficients A, B, and C with the operations of addition, subtraction, multiplication, division, and extracting of roots (here a square root).

There also exist formulae in terms of the coefficients using only a finite number of applications of $+, -, \div, \times, \sqrt[N]{\ }$ which enable one to solve third degree (cubic) and fourth degree (quatric) equations of the form

$$Ax^3 + Bx^2 + Cx + D = 0 \text{ with } A \neq 0$$

AND

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0 \text{ with } A \neq 0$$

They are not as simple as the one for solving the quadratic equation $Ax^2 + Bx + C = 0$, with $A \neq 0$ and are often not taught in secondary school courses.

For many years mathematicians sought for a similar solution for fifth and higher degree equations. There are, of course, many fifth degree equations such as $x^5 - 32 = 0$ or $x^5 + x^4 + x^3 + x^2 + x - 5 = 0$ which can be solved in terms of the coefficients and a finite number of operations of $+, -, \times, \div, \sqrt[N]{\ }$ but what we wish is a *general solution* of

$$Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F = 0$$
$$\text{with } A \neq 0$$

in terms of the permitted operations and the coefficients A, B, C, D, E, F. It was the theory needed to solve this problem, among other things, that 21 year old Evariste Galois hurriedly scribbled on the evening before he was killed in a framed duel which was the result of his amorous escapades. Galois' theory enables us to use group theory to show that it is, and always will be, *impossible* to devise a formula which will produce an exact solution of a *general* polynomial equation of degree five or higher in terms of its coefficients and a finite number of operations of addition, subtraction, multiplication, division and the extraction of roots. This does not, of course, refer to *approximate* solutions, which may be obtained to a high degree of precision, but only to *exact* solutions such as $x = -1 + \sqrt{3}$ which is an *exact* solution of the equation $x^2 + 2x - 2 = 0$. (Substitute it in and check.)

Galois' Theory also provides a proof that it is impossible to exactly trisect a general angle using only a *Euclidean straight edge*, a *Euclidean compass* and *permissible Euclidean constructions*. It is not difficult to produce excellent *approximate* trisections using these tools, nor is it difficult to produce exact trisections using additional tools or constructions, but that is not the name of the game. It's not difficult to prevent your opponent from scoring at basketball either, if you seat one of your players in the goal net, or to make a high score at bowling if you use two balls with a rope between them, but that just isn't the way these games are played.


Mathematics Around Us
Mathematics Around Us
Mathematics Around Us

WHAT DIFFERENCE DOES IT MAKE?**

(If Boys Learn Math Better Than Girls)

Elizabeth Fennema
University of Wisconsin-Madison

At the American Educational Research Association 1973 Annual Meeting, a paper was presented that reviewed 33 studies which used sex as an independent variable and mathematics achievement as a dependent variable (Fennema, 1973). The conclusion of this review was that there were no significant differences between boys' and girls' mathematics achievement before children entered elementary school or during early elementary years. In upper elementary and early high school years significant differences were not always apparent. However, when significant differences did appear they were more apt to be in the boys' favor when higher level cognitive tasks were being measured. Evidence was presented that many mathematics educators accept the belief that boys achieved at higher levels than do girls and that this belief could be based on erroneous conclusions stated in published research reports. It was suggested also that the mathematics education profession believed that sex differences in achievement are appropriate and this belief is evidenced by: 1) lack of discussion of sex differences in professional literaturê and 2) lack of program development which would enable girls to learn mathematics as well as do boys.

The author felt that the paper presented a damning indictment of the mathematics education profession (of which she is an active member and assumes her share of the blame). However, at the end of the presentation little reaction was evidenced and at a later post mortem session a colleague said to her: "I agree with the conclusions reached in your paper, but what difference does it make? Why should we teach mathematics to girls as well as we do boys or even be concerned about the issue?" It was an honest question and it deserves and honest answer. This is an attempt to provide that honest answer.

American public education holds as one of its basic tenets the belief that each child in our country should be provided with equal opportunity to learn. Although many factors have interfered with the attainment of this goal of education, no one would disagree—at least at the conscious level—with this basic belief. Indeed as we have continued to increase our understanding of what equality of learning opportunity means, deep conflict has arisen throughout the country. This conflict has been witnessed by confrontation between different groups of people and is currently being debated via the bussing issue and local versus state support of public education. However, no one is saying that any one group of children is any less deserving of education than is any other group. If one accepts this tenet that equal opportunity for learning should be provided for all children, then it follows that the 50% of the population that happens to be female deserves the same opportunity to learn as the 50% that happens to be male. Although the sex of a person may be an important factor in certain aspects of life, intellectual functioning is not one of those aspects. The responsibility for providing equal opporuntity to learn for all children applies not only to learners from minority groups and learners with special learning abilities and disabilities but to females as well.

Lip service has long been given to providing the kind of public education that permits those who complete it the opporuntity to choose an occupation in the professional world, in a trade, or elsewhere. Education in mathematics is particularly important in providing this freedom of choice. If one has a thorough grasp of important mathematical ideas,, entry into a variety of professions and occupations is opened. Without this knowledge of mathematics, access to many careers in scientific and technical areas is denied. Learning of mathematics is cumulative and is not done quickly. Therefore, if an adult is to have the choice of entering many occupations or professions, one has to have learned a great deal of mathematics. Therefore, from early childhood on girls must be encouraged to learn mathematics equally as well as do boys if a choice of how to spend their adult life is to be theirs. It is illegal to discriminate, de jure, by sex in employment. However, de facto discrimination exists if females are prohibited from obtaining the knowledge which enables them to become qualified for specific jobs.

There is an increasing economic need for girls to be prepared to enter and augment all phases of the labor market. Girls in our society are taught in a variety of ways that they will (and should) grow up, get married, have children (hopefully in that order) and be the manager of a beautiful home in the suburbs. However, the myth of the happy full-time ho usewife is displaced by the fact that today half of all adult women in the country are employed outside the home. They are employed largely in jobs which are menial and for which they have not been adequately trained.

Not only are women performing menial tasks at low levels of pay, but 85% of the people on welfare are women and their children. Many more women than men live at or near the poverty level. The

*From: *Wisconsin Teachers of Mathematics,* XXV, Winter, 1974.

number of women who live in poverty is largely partly because the training they have received has not given them skills to become economically independent. Only as women are given those skills will we be able to begin to eliminate the poor in our country. The "welfare mess" is perpetuated by these inadequately trained women.

Certainly, training in mathematics alone cannot eliminate the problems of poverty and welfare. But as girls are conditioned from an early age to think that such subjects as mathematics are unimportant or too hard, schools are perpetuating inadequate cognitive development for females which prohibits them from attaining the necessary tools for being independent adults.

Not only are there moral and economic reasons for girls to be educated in mathematics, there are also compelling mental health and "quality of life" reasons. People should be able to participate in activities which are stimulating and worthwhile. Women do not necessarily need to go outside the home to participate in their activities, but the need for women to be physically present in the house day by day is steadily decreasing. Labor-saving devices are readily available and children are full time in home only a few years of a woman's life. More couples are choosing not to have children and many women choose to remain single and to actively pursue a career. Since no one at a young age can make an intelligent choice about what their adult life will be, all children in public education must be provided with a variety of skills and knowledge that will provide options for choice. Just educating women to be housewives does not provide them with any options from which they might choose.

Providing equal opportunity for both sexes to learn mathematics is not simple. Merely giving both sexes identical curricula may not ensure equality of learning opportunity. Specialized programs may need to be developed to overcome the strong societal pressures that say it is not female to achieve in mathematics. Factors which account for the increasing differentiation of boys' and girls' performance after the onset of puberty must be identified so that positive action can be initiated which will enable girls to achieve at equal cognitive levels with boys. Ways to increase girls' willingness to pursue mathematics at the university level must be found. Mathematics educators and teachers need to become more sensitive to how their own beliefs influence their expectations and treatment of both boys and girls before the goal of providing equal opportunity to learn mathematics is met for both boys and girls.

Voltaire has said, "There are a legion of geniuses lost to the world because they were born women." In 1965 President Johnson reaffirmed this by saying, "The under-utilization of American womanpower continues to be one of the most tragic and senseless wastes of out times." Yes, it does make a difference if girls are not given the opportunity to learn mathematics and we must be concerned with the issue. We must eliminate the faulty idea that girls are inherently less capable than boys in mathematics and develop programs that not only permit but actively encourage girls to achieve at all cognitive levels as do boys. It is time to recognize the moral, social and economic responsibility of our schools to provide girls and boys equal opportunity to learn mathematics.

REFERENCE

Fennema, E. H. Sex difference in mathematics learnings: a review. *Journal of Research in Mathematics Education,* In Press.

THE DIVISION ALGORITHM — A PEDESTRIAN VIEW OF THE NUMBER LINE

James R. Boone
Department of Mathematics
Texas A&M University

The division algorithm states that for every pair of integers a and b, where $b > 0$ there exists unique integers q (the quotient) and r (the remainder) such that $a = qb + r$ where $0 \leqslant r < b$. For example: if $a = 43$ and $b = 6$, then $q = 7$ and $r = 1$ (or $43 = 7 \cdot 6 + 1$) and if $a = -38$ and $b = 5$, then $q = -8$ and $r = 2$ (or $-38 = 8 \cdot 5 + 2$). It is this algorithm that makes the "long division" process valid. We actually invoke the division algorithm in the initial step of a long division problem. In dividing 253 by 7, we may ask "How many 7's are there in 25?", but in reality the question we answer is "What is the largest number which when multiplied by 7 yields a number which is less than 25?".

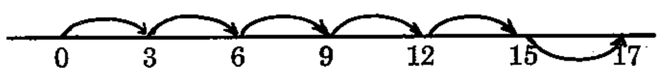
Since the number line is used extensively throughout the grades, a possible means of conveying the meaning of the division algorithm may be to consider a walk on the number line. After $|q|$ steps (backward if $a < 0$ or forward if $a > 0$) of length b, we must be to the left (less than) of a or at a, because we must step *forward* ($r > 0$) to arrive at a or we are at a ($r = 0$). This can also be considered in terms of striking arcs to the right (+) or left (—) of length b.

I visualize these principles being "played" on number lines, either at the desk by striking arcs with a compass or by steps on a number line on the classroom floor. Cuisenaire rods can be effectively used to illustrate elementary applications of these principles.

Some examples are:

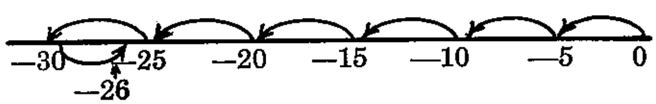
1. *Divide 17 by 3.* ($17 = 5 \cdot 3 + 2$.)

Thus 17 is 5 steps forward ($a > 0$) of length 3 and then one step *forward* of length 2.



2. *Divide —26 by 5.* ($-26 = -6 \cdot 5 + 4$.)

Thus —26 is 6 steps backward ($a < 0$) of length 5 and then one step *forward* of length 4.



3. Cuisenaire Rods.

Divide 10 by 3. ($10 = 3 \cdot 3 + 1$.)

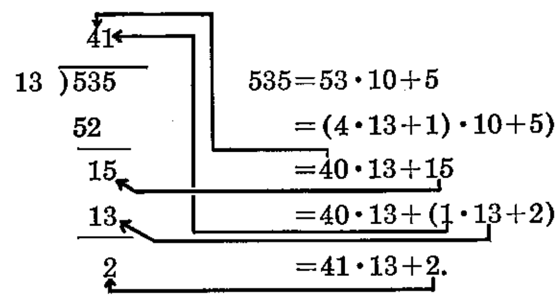
green	green	green	w
orange			

Divide 17 by 5. ($17 = 3 \cdot 5 + 2$.)

yellow	yellow	yellow	red
orange		black	

The next example illustrates the use of the division algorithm in the long division process.

4. *Divide 535 by 13.*



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