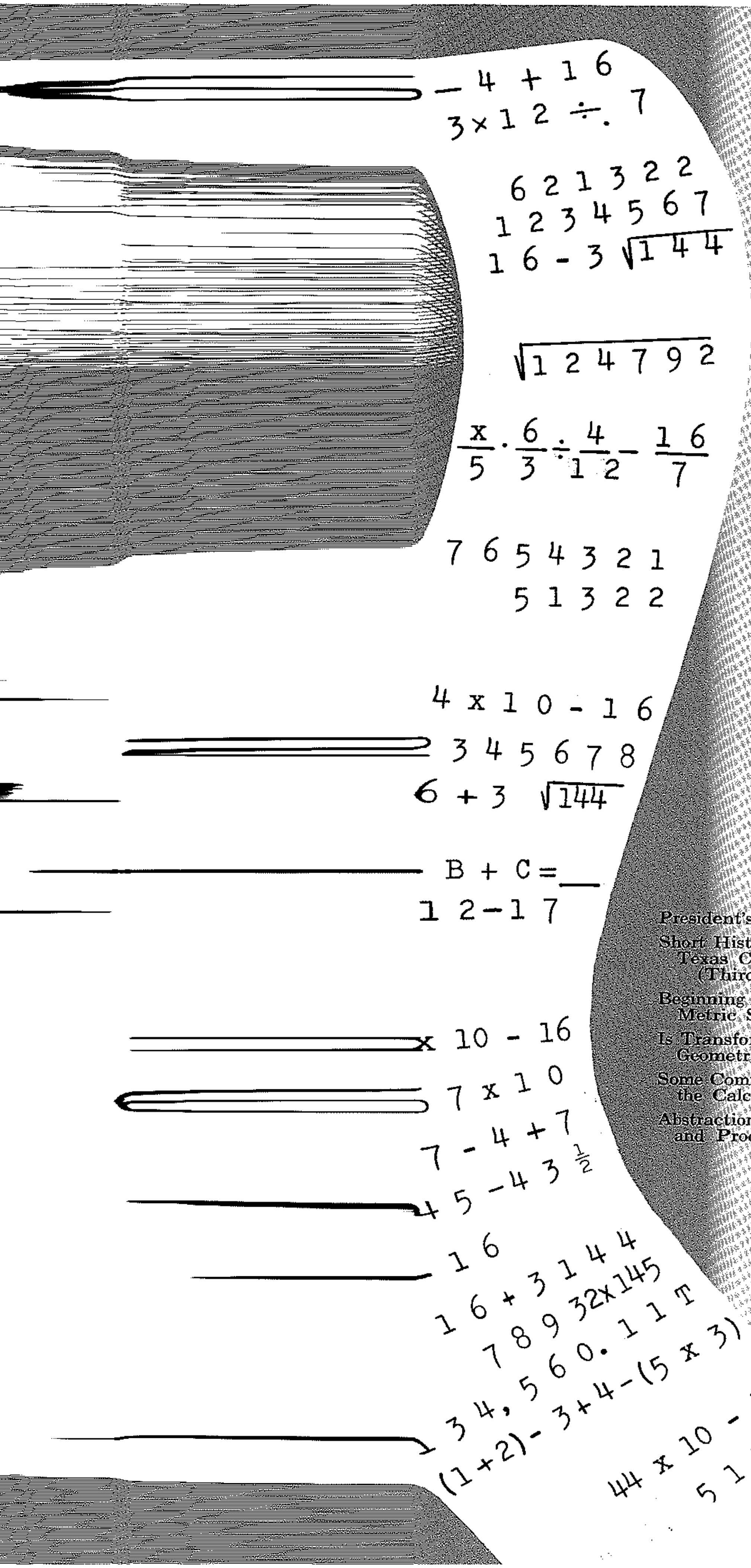

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1 Adapted fr [illegible] a speech given at the a
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lantic City, [illegible] April 1974.

countered. There is no need to provide detail about the metric system as a system; that kind of information just won't be of much value for most adults. Certainly approaching the metric system via tables of units and equivalents will mask the information that will be of practical importance.

At the same time that the metric system is introduced, however, groundwork must be laid so that parents understand how information they get will fit into the system that their children will learn. The differences between the pedagogical approaches used with adults and those used with their children should be made implicit. For example, the undesirability of teaching conversion factors to children might be pointed out.

One important consideration in developing adult education programs is the timing of instruction. It should be provided just as the need is felt by adults. If provided too early it will be washed out; and if provided too late, it will not satisfy the need. Public service groups such as Jaycees and the local Chamber of Commerce may be quite helpful to teachers in reaching a wider audience than might be met through groups such as the PTA. The involvement of the community in implementing a program of public information can significantly enhance the effectiveness of any such program.

How should the metric system be taught to children? An observation of primary importance is that the metric system is just one of the infinite variety of standard systems that could have been developed. It is the standard system that is currently accepted throughout most of the world; so as incorporated in the mathematics curriculum, the metric system as it embodies the properties of a standard measurement system should be balanced against the metric system as it shows itself in practical, real-world situations. Also important is the body of information that constitutes the metric *system.*

The metric system of measurement *is* one standard system of measurement. As such it should be put in the perspective of general concepts of the theory of measurement. That is, initial discussion should strive to develop understanding of the process of comparing physical objects with a predetermined unit and assigning to the physical object a number as a result of the comparison. Discussion of desirable properties of a system of standard units should flow from this general setting. Arbitrary non-standard units can be used to develop the concepts of precision and error and to motivate the need for widely accepted standard units, which are coherently related to each other. It is important in such a development both that measurements be assigned to objects and that objects be matched to given measurements. The correspondence between objects and measurements is in a real sense reversible, though typically only the measurement-assigned-to-object aspect has been emphasized in curriculum materials.

One activity that can be used in the study of measurement and developed with the study is estimation. Estimation with arbitrary units has the potential of fostering flexibility in making comparisons between objects. Such flexibility would make the metric system units more useful in that students would develop a better feel for sizes of objects than if estimation were never done. Estimation should of course be done as metric units are introduced, but an introduction to estimation with arbitrary units seems essential.

Estimation can take two major forms. The most common is the estimation of the measurement of one attribute of some given object; e.g., the length of a table, the area of a floor, the volume of a glass. A second kind of estimation is the identification solely by sight of an object to match a given measurement; e.g., find something 25 cm long, find something with an area of 6 m^2. For example, what do you have in your pocket or purse that has an area of 1 dm^2? If you measure the area of a one dollar bill you'll find that it provides a nonsquare model of an area very close to 1 dm^2. As a variation on this, consider the problem of finding three objects whose total length is 2 m. Such an activity forces the learner to think in terms of physical world models of measurement units. Such conceptual models are probably more useful than the concepts developed by the symbolic manipulations that typically appear in textbooks.

Practical uses of metric units in the everyday world must also be considered. Not every unit will be used equally often, so special attention must be paid to those units that must be best understood: meter, centimeter, millimeter, kilometer; liter, milliliter; kilogram, gram; degree Celsius. Those who claim that *only* the common units need to be taught, however, miss one of the essential beauties of the metric *system.* It is, after all, a system, and the simplicity is best revealed when the interrelationships among units are understood. The proper balance can be provided naturally by using practical situations for studying measurement. Common units will be highlighted in this way, and other instructional time can be used to give attention to the system as an entity.

Probably most important is the actual use of metric units in physical situations. Through measurement and estimation activities, students will develop a feel for the sizes of the units and will be better able to compare different units. As follow-up of these activities, more abstract exercises can be used, in moderation of course.

1. For each pair of measurements circle the smaller:
   a. 6m, 6000 cm
   b. 4.25 kg, 425 g
   c. 8 m^2, 800 000 cm^2
2. List the following in increasing order:
   7m, 245 dm, 6cm, 82mm

   As skill in using metric units increases, more complicated activities can be devised.
3. Make a pattern that will fold into a box without a top with a volume of 24 cm^3. How many patterns can you make?
4. Draw a triangle with perimeter of 26 cm and an area of 24 cm^2.

One other way in which the metric system can be taught, or at least practiced, is through games. The caution to be made here is that most of the games produced commercially involve only the symbolic relationships among various units. Consequently, the learning that takes place is almost certainly at a very abstract level. Unless considerable concrete experience is provided before the use of these games, the effects of the learning are likely to be washed out rather quickly. Typical commercial games can be used to provide practice at an advanced stage of learning the metric system, but too early introduction of these might put emphasis on the wrong points.

Any activity that involves measurement can be adapted for use in teaching metric units. The tremendous interest that has developed in the metric system will hopefully spark interest in making wildly creative problems, games, puzzles, and activities for teaching measurement.

Answers to pretest:

1. T
2. F (about 1 cm)
3. F (about 2½ minutes)
4. T
5. F (about 100 kg)
6. F (easy to lift)
7. T
8. T
9. F (about 70° C)
10. T

IS TRANSFORMATION IN GEOMETRY REALLY NEW?

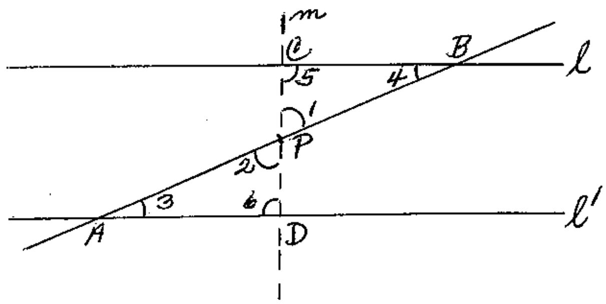
by **KENNETH CUMMINS**
Kent State University

The answer is "yes and no" and the purpose of this note is to discuss and enlarge on this reaction.

A quick "no" might come from the rather superficial thought that the not-well-defined "superposition" of Euclid used in various ways is the result of transformations but Euclid seemed not to regard it as such. His "superposition" seemed to involve getting certain given corresponding pairs of equal parts of figures to coincide *in some manner* and then to argue with the help of previous statements that the other pairs would therefore coincide and hence the figures would be congruent. On the other hand, the transformation used in geome-

try is clearcut and well-formulated — it is a one-to-one mapping onto in the space defined from one set of points to another set of points. Abstractly there is no sense of motion in the mapping but in the practice of using isometrics one thinks of reflections, translations and rotations as physical processes. Hence in the above sense we turn our hastily said "No" to a more carefully arrived at "Yes — transformations are new."

In another instance in geometry, however, there is used a strategy which is indeed close to the isometry known as rotation. Most texts which use the Legendre sequence prove the alternate-interior angle theorem (If two parallel lines are cut by a transversal then the alternate-interior angles are congruent.) by using "hypotenuse-and-angle" but



a few others have another method. P is taken as the midpoint of $\overline{AB}$ and previous theorems permit one to say that if line m is perpendicular to l then it is perpendicular to l′ also and $\triangle$s ADP and BCP are both right triangles with $\angle 1 \cong \angle 2$ by virtue of the vertical-angle theorem. Now *rotate* the $\triangle$PBC about its vertex P so that $\overrightarrow{PB}$ falls along $\overrightarrow{PA}$. This rotation (half-turn) causes B and A to coincide and line m to coincide with itself and it keeps angles 5 and 6 as right angles. Since there is at most one line from the point A (= B) perpendicular to the line m $\overrightarrow{AD}$ falls along $\overrightarrow{BC}$ and therefore $\angle 3 \cong \angle 4$[1].

It appears that the above method using rotation could be placed in the more modern setting and language of transformation approaches and would serve well. Indeed, Coxford and Usiskin use two reflections — one about m and the other about the line perpendicular to m at P — to prove this theorem and the composite of these two reflections about P is the *half-turn* used by Milne and Wenthworth.[2] We are therefore led to "No—transformations are not new" as an answer to the original question.

Perhaps one result of this study is the hope that teachers who feel reluctant to employ transformations in geometry might reflect on the fact that writers before the turn of the century were using such methods in a less sophisticated manner but with convincingly effective results. Of course, modern approaches in all the different ways to study geometry are more precise and have firmer logical underpinning, and we would expect them to be so.

1 This proof appears in Milne, William J., *Plane Geometry* (Chicago: American Book Company, 1899), p. 30. A similar one using rotation is found in Wentworth, G.A., *Plane and Solid Geometry* (New York: Ginn and Company, 1888), p. 26.

2 Coxford, Arthur F. and Usiskin, Zalman P., *Geometry: A Transformation Approach* (River Forest: Laidlaw Brothers, 1971), p. 214.

SOME COMMON ERRORS IN THE CALCULUS CLASSROOM

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Brooklyn College

Isaac Newton (1642-1727), and Gottfied Wilhelm Leibniz (1646-1716) are given credit for introducing calculus to the world in the 17th Century. The invention of analytic geometry and calculus marked the beginning of the modern era in mathematics, which had thus progressed from ancient geometry and medieval algebra. For almost 100 years, calculus was a controversial subject argued about by mathematicians, physicists and philosophers. There were struggles to assess the proper credit due each inventor as well as to establish a firm logical support for what seemed to some just a collection of compensating errors. It was Augustin Cauchy (1789-1857) who set down one of the best early treatises on the logical foundations of the calculus.

Some of the early errors in calculus that appeared in the work of even distinguished mathematicians recur in the work of the student of elementary calculus. He also has other difficulties with the concepts, some of which are very different from those studied in precalculus courses. In addition, his algebraic techniques are often so weak that the struggle to master the new concepts is undermined. It is helpful for the instructor of calculus to anticipate errors students commonly make in order to prepare plans for prevention or remedial action.

Before taking calculus, students learn the definition of "absolute value":

$$1x1 = \begin{cases} |x| \text{ if } x \geqslant 0 \\ -x \text{ if } x < 0 \end{cases}$$

but rarely apply it to situations commonly found in the calculus course. Hence students are found to write:

$$|x| + 2(x-1) + 4(x-2)$$

$$= \begin{cases} x + 2(x-1) + 4(x-2), x \geqslant 0 \\ -x - 2(x-1) - 4(x-2), x<0 \end{cases}$$

$$\text{or} = \begin{cases} x + 2(x-1) + 4(x-2), x \geqslant 0 \\ -x + 2(-x-1) + 4(-x-2), x<0 \end{cases}$$

The errors indicate that the definition has been memorized sufficiently well for purposes of finding $|-3|$, but without real comprehension of its meaning.

The absolute value function is important throughout the calculus course for illustrating concepts (such as that of a function which is continuous but not differentiable at a point), for clarifying the meaning of $\sqrt{(f(x))^2}$, and for proofs. It therefore pays to give students special practice in the use of the absolute value symbol, restating the definition as follows:

$$|Q(x)| = \begin{cases} Q(x) \text{ if } Q(x) \geqslant 0 \\ -Q(x) \text{ if } Q(x) < 0 \end{cases}$$

The absolute value symbol is an important ingredient in the definition of $\lim_{x \to a} f(x)$. It is generally recommended that this rigorous definition of limit be postponed until at least the second year in the case of the average student. However, the fundamental definition of the derivative, based on either an intuitive or rigorous definition of limit is essential for an understanding of the derivative. In order for a student to grasp the idea of average rate of change, instantaneous rate of change, and the difference between them he must spend time finding $\frac{f(x + \triangle x) - f(x)}{\triangle x}$ and its limit as $\triangle x$ approaches 0 for a variety of given functions. Unfortunately, poor support from weak algebra causes many frustrations in pursuing this goal. Consider this case:

$$\text{Let } f(x) = \frac{x + 3}{2x + 1}$$

The first stumbling block is the calculation of $f(x + \triangle x)$, notation which students find difficult to understand at first. The next difficulty is computation with fractions. There is often widespread confusion over the simplification of an expression like

$$\frac{\dfrac{x + \triangle x + 3}{2(x + \triangle x) + 1} - \dfrac{x + 3}{2x + 1}}{\triangle x}$$

Many students simply do not remember how to add fractions. Others will leave out parentheses, causing errors in multiplication and signs:

$$\frac{2x + 1 (x + \triangle x + 3) - x + 3 (2(x + \triangle x) + 1)}{\triangle x (2(x + \triangle x) + 1) (2x + 1)}$$

Results do not come out as expected, leading to acts of desperation:

$$f'(x) = \lim_{\triangle x \to 0} \frac{6x + \cancel{\triangle x} + 3 + 6x + 6 \triangle x + 6}{\cancel{\triangle x} (2x + 2 \triangle x + 1) (2x + 1)}$$

The students are much happier when they are allowed to find derivatives by shortcuts given by theorems. They master the chain-rule when it is applied to $(f(x))^n$ without too much difficulty. They put up a fuss, however, when its application is needed in implicit differentiation. The derivative of $(f(x))^n$ with respect to x is accepted as $n(f(x))^{n-1} f'(x)$ but students find it hard to understand the difference between the derivative of y^n with respect to y and that with respect to x. They must be reminded several times of the assumption that $y = f(x)$ and that $\frac{d}{dx} y^n$ requires the chain-rule.

To reduce errors in implicit differentiation, it is beneficial to construct a table as follows. Assume that $y = f(x)$ and that $x^{10} + y^{10} - 3x^3y^4 = 12.8$.

Term	Derivative with respect to x
x^{10}	$10x^9$
y^{10}	$10y^9y'$
$3x^3y^4$	$3x^3(4y^3y') + 9x^2y^4 = 12x^3y^3y' + 9x^2y^4$
12.8	0

This is helpful since it enables students to focus on each term separately, combining the results only after differentiation is completed.

Finding derivatives from given equations is hard enough; some students become really distressed when confronted with verbal problems from which they must determine their own equations. They appreciate step-by-step procedures to follow. For instance in solving a typical related rate problem, students should be encouraged to

1. Draw a diagram;
2. Write a table of information, symbolizing what is known and what is to be found;
3. Write an equation relating to the variables of the problem;
4. Differentiate according to the question posed in the table of information;
5. Substitute given values.

These steps may be illustrated by the solution of the following typical related rate problem.

A ladder 25 ft. long is leaning against a vertical wall. If the bottom of the ladder is pulled horizontally away from the wall at 3 ft./sec., how fast is the top of the ladder sliding down the wall, when bottom is 15 ft. from the wall?[1]

1) Diagram:

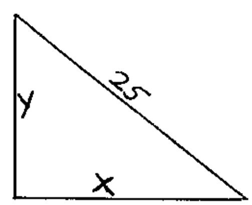


Table of information:

Let t = number of seconds elapsed
Let y = number of feet from the ground to the top of the ladder
x = number of feet from the wall to the bottom of the ladder

2) $\frac{dy}{dt} = 3 \qquad \frac{dx}{dt} = ?$ when $y = 15$

3) $x^2 + y^2 = 625$

4) $\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt} 625$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

5) Since $x^2 + y^2 = 625$

$$x^2 + 225 = 625 \text{ when } y = 15$$

$$x = 20$$

$$20\,\frac{dx}{dt} + 15\,(3) = 0$$

$$\frac{dx}{dt} = -\frac{9}{4}$$

To deter students from working mechanically, one must introduce variety into the problems. For instance, related rate problems should not always depend on right triangles; in maximum-minimum problems, setting the derivative equal to zero should not always solve the problem. Also students should check to see if their answers are reasonable.

Consider the following problem which illustrates the utility of a diagram and the need for understanding concepts rather than working mechanically.

"Find the point on the circle $x^2 + y^2 = 1$ which is nearest to (2,0). The unsuspecting student sets up this problem by using x as independent variable and obtains for the square of the distance from (x,y) to (2,0),

$$L^2 = (2 - x)^2 + 1 - x^2 = 5 - 4x.$$

Differentiating and equating the result to zero, he obtains the disconcerting expression $-4 = 0$." [2]

At this point, a graph of $x^2 + y^2 = 1$ indicates that (1,0) is the closest point and serves to remind the problem solver that $-1 \leqslant x \leqslant 1$. Returning to the expression for the derivative,

$$\frac{dL}{dx} = \frac{-2}{\sqrt{5-4x}}$$

we see that $\frac{dL}{dx} < 0$ for all values of x. Thus L decreases as x increases. The minimum value of L occurs for $x = 1$.

A rough check by differentiation would prevent the following typical error in integration. This error occurs among students working mechanically without thinking of the meaning of the integral as the limit of a sum.

$$\int_{-1}^{1} (x^2 - x)^2\,dx$$

Let $u = (x^2 - x)$, $du = (2x - 1)\,dx$,

$$dx = \frac{du}{2x - 1}$$

$$\frac{1}{2x-1}\int u^2\,du = \frac{1}{2x-1}\,\frac{u^3}{3} =$$

$$\frac{(x^2 - x)^3}{(2x - 1)^3}\Bigg|_{-1}^{1} = 0 - \frac{8}{-9} = \frac{8}{9}$$

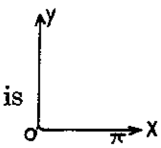
Students should be given illustrations and forewarned against other careless substitutions such as $\text{Sin}\,\frac{x}{2}$ for $\sqrt{1 - \cos x}$ without regard to appropriate sign, and degree measure for radian measure.

Textbooks often fail to emphasize sufficiently that $\frac{d}{dx}\text{Sin } x = \cos x$ and $\int \cos x\,dx = \text{Sin} + C$ for radian measure only; other formulas must be used if x represents degree measure.

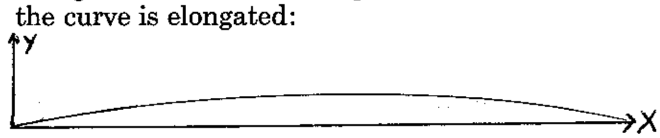
To help students to see vividly the difference between $y = \sin x$, for $x =$ measure in degrees and $y = \sin x$, for $x =$ measure in radians, have them sketch the 2 graphs on the same axes. [3] At first, students want to draw the identical graphs for both.

Computing the area under one arch of the sine curve is also instructive.

$\text{Area} = \int_0^{\pi} \text{Sin } x\,dx = 2$. The graph is



Using the *same* units to represent degree measure, the curve is elongated:



$$\text{Area} = \int_0^{180} \text{Sin } x\,dx =$$

$$\frac{180}{\pi}\int_0^{180} \frac{\pi}{180}\,\text{Sin } x\,dx = -\frac{180}{\pi}\,\text{Cos } x\,\Bigg|_0^{180}$$

$$= \frac{360}{\pi}$$

The latter computation emphasizes the point that differentiation and integration formulas for trigonometric functions in degree measure and for trigonometric functions in radian measure are different.

Students enjoy L'Hospital's Rule so much that they like to use it everywhere. They sometimes forget that they have simple methods for finding such limits as

$$\lim_{x \to 0} \frac{7 + 3x}{4 - 3x} \quad \text{or} \quad \lim_{x \to \infty} \frac{x - \text{Sin } x}{x}$$

and attempt to differentiate numerator and denominator, contrary to the hypotheses of L'Hospital's theorem.

Although students develop a great affection for L'Hospital's Rule, they seem to see little need for it at first. Isn't $\frac{0}{0} = 1$? Some say $\frac{0}{0} = 0$. It is convenient to have a supply of quick convincing illustrations to show how $\frac{0}{0}$ can arise, and may represent any number that we choose.

$$\lim_{x\to 0} \frac{2x}{x} = 2; \quad \lim_{x\to 0} \frac{3x}{x} = 3, \text{ etc.}$$

Similarly, a variety of possibilities can be shown for each of the forms

$$\frac{\infty}{\infty},\ \infty - \infty,\ \infty^{\circ}.$$

If our students mistreat infinite series, we ought not to be too surprised, but we should be prepared. The history of the calculus indicates how natural such errors are. A few centuries ago, mathematicians were arguing about

$S = 1 - 1 + 1 - 1 + 1 - 1 + \cdots\cdots$

Some said $S = 0$ since $(1 - 1) + (1 - 1) + \cdots = 0 + 0 + \cdots$

Others countered with $S = 1$ since $1 - (1 + 1) - (1 + 1) - \cdots = 1 - 0 - 0 \cdots$

Leibniz seemed to think that, since

$S = 1 - (1 - 1 + 1 - 1 +) = 1 - S,$

$2S = 1,$

and $S = \frac{1}{2}$.

Even Euler "held that from $\frac{1}{(1 + 1)^2} = \frac{1}{4}$ one could conclude that $1 - 2 + 3 - 4 + 5 \cdots = \frac{1}{4}$." (4)

The study of calculus should be an exciting adventure. The mathematics is fascinating and has extensive applications. To maximize success and minimize errors, however, the instructor should emphasize the meaning of each concept and require that students justify the mechanics that they use. He should encourage students to check to see if his answers are reasonable. The patient instructor who also gives his students a good review of the necessary algebra and trigonometry will be rewarded by the accomplishments and appreciation of his students.

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ABSTRACTION: PROCESS AND PRODUCT

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School mathematics curricula, instructional practices, and student achievement are again the object of widespread public attention. Programs that are second and third generation products of SMSG, UICSM, UMMaP, and other "New Math" development projects are facing criticism for any or all of the following sins:

—too much emphasis on concepts, too little practice with skills;

—to much emphasis on theory; too little training in practical problem solving;

—too much emphasis on symbolism and deductive proof, too little attention to number and space intuition;

—too much emphasis on abstract structures, too little experience with concrete models of ideas;

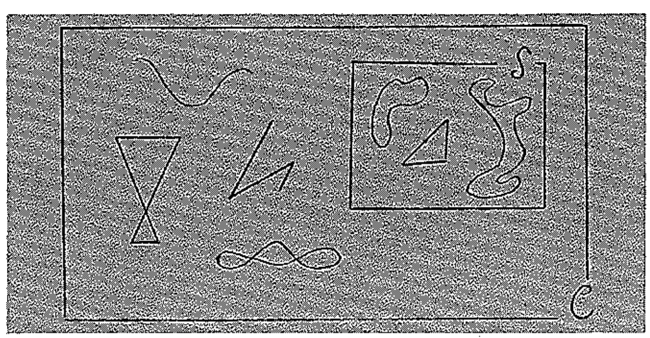
—too much emphasis on discovery learning, too little use of needed drill.

These criticisms strike a respondant chord with many teachers and students. But in the movement to less formal, more practical curricula, the important payoff from mathematical *abstraction* is being lost — victim of guilt by association with genuine excesses in *rigor, symbolism,* and *deduction.* The loss is unfortunate because,

(1) A statement of fact, a process for organizing information, or a problem solving procedure is mathematical only if it is abstract.

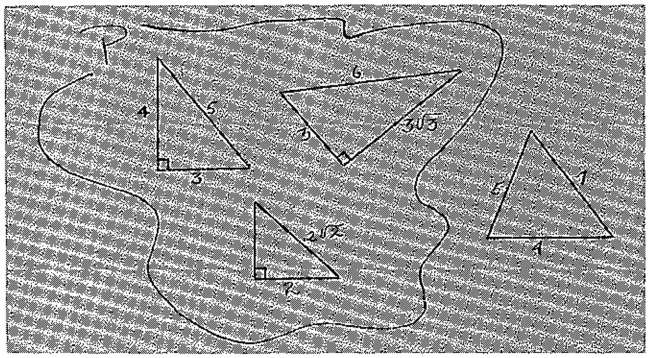
(2) Teaching students abstract ideas and the process of making abstractions is an important part of school mathematics instruction — it can be fun and can lead to exciting problem solving payoff.

In a simple-minded way, abstracting is the process of recognizing that two apparently different situations actually have similar properties. In fact, the power of a mathematician is the frequency with which he can say "I've seen a problem very much like this before." Abstraction occurs when one formulates a definition.

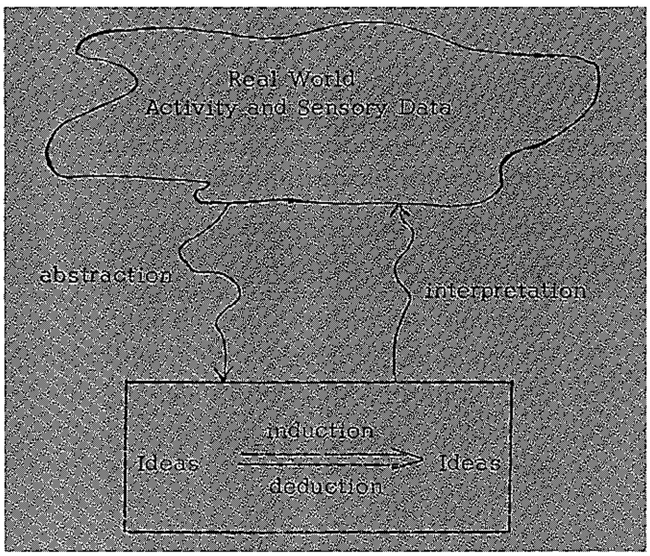


The curves in set $\mathcal{S}$ are simple and closed.

Abstraction occurs when one detects a theorem in geometry.



$a^2 + b^2 = c^2$ if and only if C is a right angle. Abstraction occurs when one solves a "real world" problem.



Mathematically educated students must have learned the abstract concepts and structures most frequently seen to model reality, and they must have facility in recognizing the hookups between ideas and real-life situations often over-run with perceptual noise. It might be argued that to achieve this goal teachers need only to create frequent word problem experiences and inductive teaching (pattern searching is abstraction). These pedagogical strategies are essential. But if the set of mathematical structures studied includes only the real number system and standard euclidean geometry, students too often learn structural rules by rote (particularly in algebra). New systems which violate the seemingly universal number properties (for instance, matrix multiplication) are never quite accepted as legitimate mathematics.

Example 1. Ask a junior high math class "What makes the following statement true: $7+(x+3)=(7+x)+3$?" and the response will include all the new math buzz words from 'soshative' through 'communitive" — and at least one "if you add 'em up you get the same answer." Each of us has at times felt frustration with the difficult connection between names and properties or between properties and proper problem solving strategies.

Why do we bother stressing properties of the number system? Better understanding, retention, and transfer are the common answers. Why do students have so much trouble keeping the properties straight? Not because the properties are abstract, but because they have not been taught in a sufficiently abstract context! One promising antidote for this difficulty is to present many realizations of each number system property.

Among plausible, but non-standard, operations on numbers, one of the most enlightening is 'averaging'

$$a * b = \frac{a+b}{2}$$

Begin with a pattern search contest of "guess my rule,"

$$6 * 4 = 5$$
$$12 * 4 = 8$$
$$9 * 3 = ?$$
$$3 * 9 = ?$$

$3 * 9 * 7 = ?$ — $(3*9)*7$ or $3*(9*7)$

$$\square * 3 = 3$$
$$\square * 11 = 11$$

For a real challenge, try to determine whether ordinary $\cdot$ and + distribute over * and what each would mean in terms of scaling and/or curving test scores.

$$a \cdot (b*c) = (a \cdot b) * (a \cdot c)$$
$$a + (b*c) = (a+b) * (a+c)$$

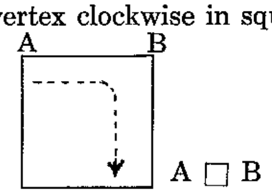
Other practical and structure revealing operations you can use include—

Max: $a * b$ = larger of a, b
Min: $a * b$ = smaller of a, b
1cm: $a * b$ = least common multiple of a, b

To push comprehension of the abstract structure patterns one step farther, try these operations on pairs of points in a plane—

A * B = midpoint of AB

A □ B = third vertex clockwise in square ABCD



You'll be delighted at the speed with which students conjecture non-associativity of "midpoint", based on an intuitive isomorphism with averaging. Furthermore, you'll have students jumping out of their seats to play 'guess my rule' with operations of their own creation. By seeing examples and counter-examples of the number system properties, students will sharpen their insight into application of those patterns.

Example 2. Ask most laymen to define a straight line, and you will usually hear "it's the shortest distance between two points." Despite the best efforts of secondary school geometry instruction, the respondent will *very* seldom reply "it's an undefined term."

Aside from the fact that a line is most certainly not a distance (number), in our practical life space today there are few cases in which the shortest path between two points is a straight line. To understand this statement, one must move to a more general or abstract notion of distance.

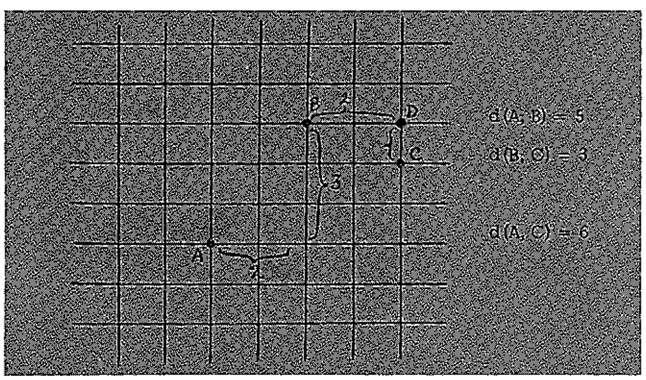
If S is a set and d is a mapping from SxS to the real numbers such that for any a, b, c in S

1) $d(a, b) > 0$ $\quad$ $(= \text{iff } a=b)$
2) $d(a, b) = d(b, a)$
3) $d(a, c) < d(a, b) + d(b, c)$

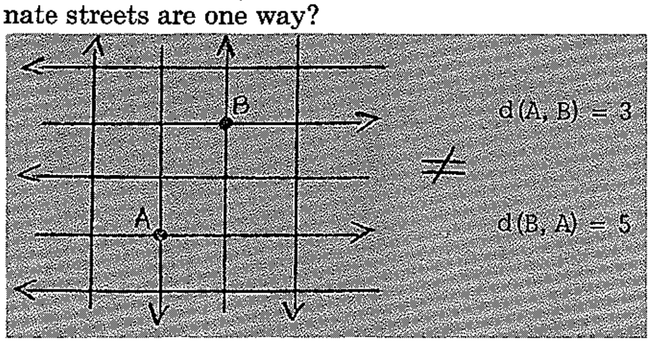
then (S,d) is called a *metric space.*

The variety of measurement procedures that fit this pattern is astounding. For instance, a suburbanite who commutes to work might measure "distance" by time or mileage or cost. An airline traveller would probably consider time or cost — seldom mileage. The telephone company measures distance and then rates by a combination of euclidian distance and facilities for carrying calls (a New York to Dallas call might actually be routed through Los Angeles). For several of these non-standard metrics, one or more of the conventional metric space properties fail to hold. This incongruity can lead to intriguing and informative classroom explorations.

For openers, try the familiar taxicab metric on a city street grid:



This rule obeys all desirable metric properties. (Note, however that d(A, D) = d(A, B) + d(B, D) A, B, D collinear.) But what if alternate streets are one way?



For sheer simplicity, and not total impracticality, investigate the "telephone connection" metric with rule:

$$d(A, B) = \begin{cases} 0 \text{ if } A=B \\ 10 \text{ if } A \neq B \end{cases}$$

It arises from the fact that 10 digits will dial any phone in the U.S.; it takes no time to talk to yourself. Most familiar metric rules hold.

Example 3. One of the main "new math" recommendations for change in school algebra instruction was unification of manipulative techniques around fundamental group and field properties. If students acquire rudimentary understanding of elementary group theory, the algebra of solving equations will fit together beautifully.

In any group (G, *) if a^I represents the inverse of a, then $a * x = b$ iff $x = a^I * b$.

This basic pattern goes to work as follows

1) $a+x = b$ iff $x = -a + b$
2) $a \cdot x = b$ iff $x = \frac{1}{a} \cdot b$
3) To solve $ax + b = c$ combine (1) and (2).
4) For vectors in a plane
   $(a,b) + \vec{\gamma} = (c,d)$ iff $\vec{\gamma} = (-a, -b) + (c,d)$
5) If $f(x) = \frac{9}{5}x + 32$, then $f^{-1}(x) = \frac{5}{9}(x-32)$

   To solve $f(x) = 68$, look at $f^{-1}(68)$.
6) To solve the linear system $\begin{cases} 3x + 2y = 1 \\ 7x + 5y = 2 \end{cases}$

   Look at matrices $M = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

   and $M^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$.

$$M \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{iff} \begin{bmatrix} x \\ y \end{bmatrix} = M^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Conclusion—The process and the products of abstraction are essential aspects of mathematical thought. The few illustrations given above demonstrate that abstraction need not be synonymous with sterile formalism. The search for abstract ideas opens up pedagogically exciting worlds of creative exploration, at the same time casting revealing light on the structure of specific concrete systems of number and space.

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