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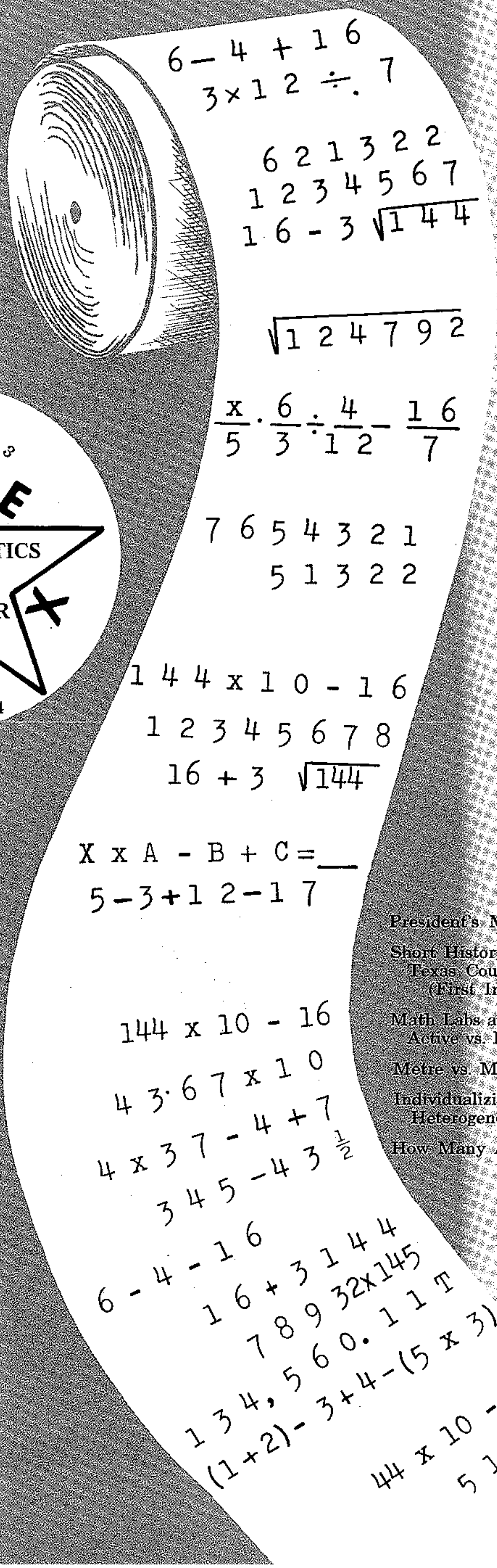

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TEXAS MATHEMATICS TEACHER is the official journal of the Texas Council of Teachers of Mathematics. The views expressed are the contributor's own and are not necessarily those of the publisher or the editor. All manuscripts and correspondence about this publication should be addressed to Mr. J. William Brown, *Texas Mathematics Teacher*, 100 So. Glasgow Drive, Dallas, Texas 75214.

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President's Message

How competent are you in performing the responsibilities assigned to you? Where did you acquire these competencies? Colleges are being asked to describe courses in terms of competencies. This is becoming possible because competencies in the various job classifications are being determined. The Commission, a group of educators appointed by the State Board of Education, is working on competencies for teacher education programs. Each professional organization has been invited to submit competencies in the particular subject area the organization represents. This invitation has come to us as mathematics teachers. The National Council of Teachers of Mathematics has published a document which thoroughly covers the subject in a booklet entitled, "*Guidlines for the Preparation of Teachers of Mathematics.*" This information will be compiled in the required format and submitted. We would welcome input from you, of course. If you have information you would like to submit, please send your reactions and suggestions to me immediately.

The dates have been set for the fall mathematics conference co-sponsored annually by Texas Association of Supervisors of Mathematics, Texas Council of Teachers of Mathematics, Texas Education Agency, University of Texas, Department of Mathematics, and University of Texas, Department of Mathematics Education. The conference will be held in Austin at the Driskill and Stephen F. Austin Hotels November 21-23, 1974. Your response last year was tremendous. This year there will be more section meetings and workshops scheduled for each of the designated times. Additional meeting rooms will be utilized to accommodate those who attend. Your early registration will greatly enhance our planning. The steering committee, composed of a representative from each of the co-sponsoring organizations, has named the following chairmen:

General Chairman.............DR. BILL GUY
University of Texas Mathematics Dept.

Corresponding Secretary....DR. ALICE KIDD
Texas Education Agency

Program Co-Chairmen...........ANNE WOOD
Spring Branch ISD
DR. HEATHER CARTER
University of Texas, (Department of Mathematics Education)

Facilities Chairman......MARVIN VESELKA
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Austin ISD

Registration Chairman..DR. IRENE ST. CLAIR
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Publicity Chairman..........CHUCK CARSON
Pasadena ISD

Commercial Exhibits Chairman
THELMA HAMMERLING
Houston ISD

Meal Functions Chairman
MAXINE SHOEMAKER
Austin ISD

NCTM Materials Chairman....MADGE SIMON
Gregory-Portland ISD

Pages Chairman..Austin Council Representative

This annual state mathematics meeting will be called the Texas Mathematics Conference (TMC) rather than CAMT as formerly named. Make your plans now to be one of over a thousand mathematics teachers who will convene in Austin Nov. 21-23.

We believe that you enjoy and look forward to the *Texas Mathematics Teacher* each publication date. You may know that printing costs are increasing as is everything else. The cost of printing the January journal was twice as much as that of the October journal. Our treasurer has indicated that our expenses for this year will exceed our income by more than $500. We want to continue our communication across the state through the journal. In order to do this additional funds must become available. It is necessary, therefore that the dues be increased to cover the cost of the journal. Your executive board will have an announcement regarding this at the annual meeting on November 22, 1974.

Suggested constitutional revisions were inserted in last issue. Please read these and be ready to react at TCTM business meeting in November.

What will math be in the 1980's? The Governor of Texas asked Texas Association of Supervision and Curriculum Development to form a task force to project education for the 1980's as they see it. Each professional organization has been invited to suggest its views. We solicit your thinking so that we may pass these ideas on to the TASCD committee. Please respond to me by June 15.

The year has almost gone! We hope it has been a good one for you and that you have seen progress with your students. Have a good summer!

—Shirley Ray

SHORT HISTORY OF TEXAS COUNCIL OF TEACHERS OF MATHEMATICS

(FIRST OF THREE INSTALLMENTS)

For many years the organizations on the state level for teachers of mathematics and arithmetic struggled to keep going. It was only through the tireless and unrelinquishing efforts of Miss Elizabeth Dice and Mrs. Lorena Holder of Dallas and Miss Mary Ruth Cook of Denton that the Mathematics Section and the Arithmetic Section survived.

These three teachers and many others felt that the brief sectional meetings during the annual conventions of the Texas State Teachers Association were unsatisfactory because they were impersonal. They gave no opportunity for teachers to become acquainted and to share in teaching experiences on the various teaching levels. Also, the nomination and election of officers for one-year terms provided no continuity for the organizations.

Upon the invitation of the Arithmetic Section of T. S. T. A. during the annual meeting in Houston in 1951, the Mathematics Section voted to unite with the Arithmetic Section in forming a mathematics group to become affiliated with the National Council of Teachers of Mathematics. A committee composed of Mrs. Lorena Holder from the Mathematics Section, Miss Lois Averitt from the Mathematics Section, and Miss Mary Ruth Cook, the State Representative of the National Council, was appointed to draw up a constitution for the new organization and to present it for adoption at the annual meeting in 1952.

In a joint meeting of the Arithmetic and Mathematics Sections in El Paso, November 28, 1952, the constitution was adopted, and the Texas Council of Teachers of Mathematics was organized. Twenty-eight teachers became members of the new organization that day. Officers elected for the first year were President, Miss Joyce Benbrook, Houston; First Vice-President, Mrs. Lorena Holder, Dallas; Second Vice-President, Miss Lois Averitt, Denton; Secretary-Treasurer, Miss Rebekah Coffin, El Paso; Parliamentarian, Miss Nathalie Dinan, Beaumont; Editor, Miss Pat Copley, Dallas; and State Representative for the National Council, Miss Pearl Bond, Beaumont. Dr. Walter Carnahan from Purdue University addressed the first meeting of the organization.

The second meeting of the Texas Council of Teachers of Mathematics was held in Dallas, November, 1953. Dr. William Gager of the University of Florida was the speaker for the joint meeting. Officers who served during the second year were President, Miss Lois Averitt, Denton; First Vice-President, Mrs. Lorena Holder, Dallas; Second Vice-President, Miss Fay Noble, Sherman; Secretary-Treasurer, Miss Rebekah Coffin, El Paso; Parliamentarian, Miss Nathalie Dinan, Beaumont; Editors, Miss Alice McCall and Miss Odessa Drake, Beaumont; and State Representative to the National Council, Miss Pearl Bond, Beaumont.

The third annual meeting was held in Fort Worth in 1954 with Dr. Esther Swenson of the University of Alabama as speaker. Officers elected for the following year were President, Mrs. Lorena Holder, Dallas; First Vice-President, Miss Margaret Kennerly, Houston; Second Vice-President, Miss Fay Noble, Sherman; Secretary-Treasurer, Mrs. Maurine Aldrich, El Paso; Editor, Miss Izetta Sparks, Denton; Parliamentarian, Miss Nathalie Dinan, Beaumont; and State Representative for the National Council, Miss Pearl Bond, Beaumont.

In San Antonio in 1955, Dr. Ida May Heard of Southwestern Louisiana Institute gave the address. Officers elected for 1956 were President, Miss Fay Noble, Sherman; First Vice-President, Miss Margaret Kennerly, Houston; Second Vice-President, Mr. Arthur Harris, Dallas; Secretary-Treasurer, Mrs. Maurine Aldrich, El Paso; Editor, Miss Izetta Sparks, Denton; Parliamentarian, Miss Nathalie Dinan, Beaumont; and State Representative for the National Council, Miss Pearl Bond, Beaumont.

The fifth annual meeting was held in Houston in 1956 with Dr. Milton W. Beckmann of the University of Nebraska as speaker. Officers elected for 1957 were President, Miss Margaret Kennerly, Houston; First Vice-President, Mrs. Mozelle Schulenberger, Cleburne; Second Vice-President, Mr. Arthur Harris, Dallas; Secretary-Treasurer, Mrs. Dorothy Woodley, El Paso; Editor, Mr. Kenneth Mangham, Midland; Parliamentarian, Miss Nathalie Dinan, Beaumont; and State Representative for the National Council, Miss Ella Porter, Houston.

At the sixth annual meeting of the Texas Council in Dallas in 1957, Dr. Howard Fehr of Columbia University gave the address. Officers elected for the following year were President, Mr Arthur Harris, Dallas; First Vice-President, Mrs. Mozelle Schulenberger, Cleburne; Second Vice-President, Mr. Keene Van Orden, San Angelo; Secretary-Treasurer, Mrs. Dorothy Woodley, El Paso; Editor, Mr. Kenneth Mangham, Midland; Parliamentarian, Miss Nathalie Dinan, Beaumont; and State Representative for the National Council, Miss Ella Porter, Houston.

In six years the Texas Council of Teachers of Mathematics has become an influence upon many

Texas teachers and has begun to fulfill the purposes for which it was organized. It has grown from the original twenty-eight members to an organization of more than three hundred members. In 1954 the Constitution was amended to meet the needs of the growing organization. Each year a delegate has been sent to the spring meeting of the National Council of Teachers of Mathematics. In October, 1958, the Texas Council acted, for the first time, as one of the co-sponsors for the annual Conference for the Advancement of Science and Mathematics Teaching which was held in Austin.

These many activities tend to unite the mathematics teachers of Texas into one group with a single purpose—the improvement of mathematics teaching in our state.

MATH LABS AND LEARNING PACKETS— ACTIVE VS. PASSIVE LEARNING

By
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Elizabeth A. Cunningham
San Jacinto Intermediate School
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With recent talk of "gasoline shortage," "meat shortage," "wheat shortage," and "energy crisis," teachers are discovering a "motivational crisis," or "motivational shortage" in the classroom. The "energy crisis" is in the able hands of our lawmakers in Washington, but a "motivational crisis" can only be handled by educators.

The mathematics staff of San Jacinto Intermediate School in Pasadena, Texas, have found themselves in the midst of a "motivational crisis" and are meeting the challenge of overcoming the problem. Working on the premise that children are visual rather than audio learners, instruction takes place on three levels: math lab, learning packets, and computer assisted instruction. Following are brief discussions of the programs at San Jacinto.

Math Lab

The math lab at San Jacinto was established as a motivational tool. The lab is a place where visual learning is the rule. The student must see and touch the items in the lesson.

Partial funding for this project came from a $400 grant from the school district for "new teaching strategies" proposals. (Programs like this help motivate teachers.) The rest of the project was set up on resources available in the school.

The terminal objective of the math lab is in the affective domain, how the student feels. The students at San Jacinto will become more motivated to study math. They will show this increased motivation by:

1. independent research and investigation.
2. individualized instruction through the use of the math learning center and teacher-made learning packets.
3. instructional activities through games and puzzles.
4. "student helping student" reinforcement.

If the students show a 50% increase in motivation as determined by an attitude survey at the end of each trimester, the program will be considered a success.

The physical make-up of the lab is simply a classroom with tables and chairs instead of desks. Tables allow students to work independently, in pairs, or in small groups without having to move furniture. The following is a brief inventory of supplies and equipment for the lab:

1. Audio-Visual Equipment
2. Books
3. Metric Equipment
4. Geometric Equipment
5. Learning Packets
6. Games and Puzzles
7. General Teaching Supplies

Learning Packets

Another strategy is the learning packet. These are of two types: commercial and teacher-made.

The commercial packets we have used extensively are Individual Mathematics Programs (IMP). The students were given a test, Test in Achievement of Basic Skills (TABS), at the beginning of the course to determine which of the 64 objectives on the TABS test the student could already meet. Those objectives that were not successfully completed by the student were the only objectives that the student needed to work through in the corresponding packet in IMP.

The IMP packets consist of four main parts: pre-test, five lessons, self-test (a review of the packet), and post-test. If the student makes 90% on the pre-test he does not need to go through the pack. If the student achieves less than 90%, he completes the pack and takes the post-test. Then he proceeds on to the next objective. These packs can be used throughout the entire course.

The teacher-made packs are varied in length. Some are for one concept (such as reducing fractions) and some are for a whole unit (such as a two week geometry unit). They have much the same format as IMP. There are pre-tests, explanations, worksheets, and post-tests.

The student reaction for the most part has been favorable. They seem to like the individualized packs better than textbook materials and lectures. Grades have improved because students can work at their own rate and not be under pressure to complete all of the assignment in one clas period.

Computer Assisted Instruction

There are four aspects to computer instruction which can be used. The Pasadena School District operates a Hewlett-Packard Time-sharing system. San Jacinto operates with one teletype terminal.

The first aspect is the Mathematics Drill and Practice Program. This program allows each student to be enrolled in a math skills program with six, twenty-four block modules on the computer. After completing module six, block twenty-four, the student is automatically signed off. Obviously, with only one terminal, not all can participate in this phase of computer instruction, but such alternatives as follow are available.

A second available aspect is the Instructional Dialogue Facility. Using this facility a teacher can prepare a lesson with instruction, question, and answer capability. This facility is similar to programmed learning.

The third aspect is the system library. It is used daily for basic skills drill. Each sixth, seventh and eighth grade student may choose either addition, subtraction, division or multiplication drill. Some eighth grade students use a signed number drill. Algebra students use programs which generate equations to be solved.

In the fourth available aspect the algebra students are introduced to BASIC programming in the eighth grade. Computer programming is taught as an extra activity before and after school.

Conclusion

Motivation is the key to learning. Children learn what they want to learn. It is the goal of the San Jacinto math program to continually strive to offer a program aimed at overcoming the "motivational crisis."

An Open Letter to Mathematics Educators

METRE vs. METERS

by Chuck Allen, Director N.C.T.M.

Instructional Specialist for Area D, Los Angeles City Schools

A METRIC AMERICA is a decision whose time has come. The MOVE TO METRIC is a move that can no longer be denied. For once, the public schools have a chance to be ahead of this move. We have the chance to do extensive in-service and pre-service education before the move is made. Will we seize this chance?

Battle lines are rapidly being drawn. PURE SI OR NOTHING! METRE VS. METER! KILOGRAM VS. GRAM! MASS VS. WEIGHT! DEKA VS. DECA! dam VS. dkm! These many small squirmishes only provide fuel for those not ready nor willing to change. They encourage some to wait and do nothing until the experts agree. The squirmishes justify those who will take a little of the new and a little of the old. They serve to placate those who never learned the old but find the new impossible.

Have we forgotten so soon the trials and tribulations of our move to the NEW MATHEMATICS? Don't we recall the stands on NUMBERS VS. NUMBERAL? (I still don't remember whether the catsup was on the bottle or the name was on the bottle.) Do we want another bout with the PRECISION OF LANGUAGE group? Is measurement a tool to serve us or are we tools to serve measurement. Don't spellings change with the years? Isn't the measurement on the street (a liter of gas) less formal than measurement in the science lab. (a cubic decimeter of water at 3.9°C)? Must measurement continue to be something out of reach of the masses?

Having waded through the decisions of the General Conference of Weights and Measures (CGPM), I am left numb. Our current approaches offer little in the way of improvement. The MOVE is threatened!

This is not a proposal that all the churches stop their bickering and unite into ONE BIG BAPTIST CHURCH! Rather, it is a cry for some reasonable and realistic approach to a monumental task. Academic students will succeed in spite of and/or because of what we do or do not do. My concern is for the non-academic student. We must take time to consider the masses of students who are leaving our high schools without saleable skills in reading, writing, spelling, or computing. Let the Science Major make his reports in PURE SI! Let the international papers be written in PURE SI! Let the Universities conduct their classes with PURE SI! But, for goodness sake, let the man on the street have a measurement system that he can understand and use! Let the girl in homemaking, the boy in the shop class, the lad on the athletic field, and the parent in Adult School all study a measurement system that is consistent and practical in their world!

(1) METRE VS. METER? We are an English-speaking Nation so let's be an English-spelling Nation. METER!

(2) KILOGRAM VS. GRAM? Multiples are taught long before fractionals so let's have all basic units without prefixes. GRAM (basic)!

(3) MASS VS. WEIGHT? We have no choice here. People will continue to weigh rather than amass. WEIGHT!

(4) DEKA VS. DECA? and dkm vs. dam? Here, I have no preference. Let's be consistent though. If DEKA, then dkm! If DECA, then dam! (I do like the tie in with decathlon and decade.)

(5) PURE SI VS. NOTHING! Let's do have PURE SI INSTRUMENTS, PURE SI STANDARDS, AND EVEN PURE SI SPELLING ON SCIENTIFIC PAPERS. BUT, for the man on the street, let's have a MODIFIED SI! Let him buy a LITER of gas, a kilogram of coffee, and a ticket to see a football game on a field 100 yards (oops!) 90 meters long!

Many changes have occurred since our move to "NEW MATH" in 1959. Parents have demanded and received PARENT POWER. Students have discovered STUDENT POWER. Even teachers have struggled to be recognized as the authority on teaching as they gained TEACHER POWER. These groups will no longer sit idly by and have changes shoved down their throats. They cry out for involvement. In one loud chorus they sing, "TAKE US ALONG." To do less would be to write the future best seller.

WHY JOHNNY CAN'T MEASURE!

Let America adopt PURE SI. Let's use NBS Special Publication 330 in our academic classes. But, for the little ones just learning to read, let's use a modified SI. Let's make all their "er" sounding words have the same spelling. Let them learn meter, liter, theater, and center at the same time. K-12 Articulation is another change that is rapidly growing since our last move. What about the shop teachers and the business teachers who can no longer cover our classes since the new math.? Let's articulate with them also. Give them the NBS Publication, *WHAT ABOUT METRIC?*

The decision is with us. The move cannot be denied. The chance is ours. LET'S TAKE THEM ALONG!

INDIVIDUALIZING MATH IN A HETEROGENEOUS CLASS

by Bill Beckman
University City Missouri

Yes, it can be done! It has been done! It is being done now! Individualization can be accomplished by the average teacher with a full sized class. The problem is one of focus and preparation.

The teacher must move the focal point of the classroom from himself to each individual youngster in the class. Every member of the class becomes a participant, not a passive recipient. The students are helped by the teacher in brief and frequent individual help sessions. They are helped, and give help to their fellow students. The class is active, noisy, alive. Youngsters are conferring with peers, working on their own, looking at references suggested by the teacher. The children are the center of activity in an individualized classroom. The teacher becomes a resource, a helper, a fellow learner.

But before the student sees the material, much work, thought, and organization has gone into the unit. All this preparation by the teacher helps to keep all levels of student abilities functioning at a maximal level of achievement. The bright youngster is allowed to move through the materials with relative freedom. Once a youngster has displayed an obvious understanding of the basic material usually by successfully completing a testing program. A rerun is usually in order at a higher level. Much more thinking and a lot less doing!

The average youngster moves along getting help when needed. (A knack is soon developed in spotting the kids who are struggling with a particular task.) Several sources—textbooks, workbooks, dittos, games, puzzles, tapes (both voice and video) are ideally available to be sure that this group remains a collection of individual members and are treated as such by the teacher. The students who continually have trouble are invariably *reading* problems. Materials for these youngsters should be as non-verbal as possible. Many commercial publishers and NCTM publications have been devoted to non-verbal approaches to learning mathematics. Kits, puzzles, games, tactile devices are the best way, I have found, to individualize with this group.

But let's look at a specific set of achievements by a specific teacher. The whole idea was first attempted after considerable urging by Dr. Ralph Klines, a professional innovator. It was begun with a seventh grade class teaching fractions to a totally heterogeneous class. I simply took the textbook and broke it up into thirty individual units of work with tests spaced in the units to evaluate progress more often. The presumption was that all youngsters could handle the material if it was broken down into small enough units. The students corrected their own exercises while the teacher controlled the testing. This program was followed for three years with some revisions made with dittos, to clarify some perceived problems. Deadlines were established after the first year as it was found that about one-third of the students could not function without some pressure to get work done at a specific time.

It was during the three years that I also introduced the varied difficult assignments. If the basic assignment was beyond the skills of an individual, then background work and specialized assignments would be made to allow the student successful experiences while he was learning basic skills.

I found that this seeking a level of competency presented some self image problems since not many junior high students are willing to admit that they do not know their multiplication tables or how to divide. Here various motivational devices become useful and the sources mentioned above and those listed in the bibliography will be helpful.

Also, I tried the technique of allowing low achieving students to go into the elementary schools and teach third and fourth grade kids their multiplication tables and thereby learn the tables themselves. The success and failure of learning experiences rests on the student's willingness to work with material that he should know at that particular level and getting him to admit he does not know enough. An experienced teacher can impose his will on such students, but most must resort to various devices to get students to work on remedial material. Self image and a restoration of pride and accomplishment seem to be the key. Read Mr. Bill Page's reports in bibliography (1, 2).

Finally through workshops and exposure to the UNIPAC idea from the Kettering Foundation I. D. E. A., the constructing of Behavioral Objectives for each of the units of work was started. The UNIPAC itself is a very useful tool but its production is a tedious and time consuming task. The key is for each teacher to select a small topic and write a unipac until a predetermined unit is covered. This can be done with as broad a spectrum of teachers, each teacher doing his/her part creating a bank of learning packages use ful to all.

Let's look at a particular learning package with pre-test, behavioral objectives, recycling for reinforcement and enrichment and post test. It is best to begin with a short topic. I began with fractions. Following is a part of this package covering just equivalent fractions.

WE WILL STUDY EQUIVALENT FRACTIONS

1. You will know the definition of equivalent fractions and be able to express the definition in your own words if requested to do so.
2. You should be able to list three fractions that are equivalent to any given fraction.
3. You should be able to use the property of one to convert one fraction into a desired equivalent fraction.

Now see the teacher for a pre-test to see what you already know.

I. Using your text read page 197 and work exercises on pages 197-198, numbers 1-5. Check your work with the teacher.

A. Optional BBM I—read pages 266-273
Work exercises on page 267, number 1-9; page 269, numbers 1-4; page 271, numbers 1-2; page 273, numbers 1-8. Check your work at teacher's desk.

B. Optional BBM II—read pages 244-253
Work exercises pages 244, numbers 1-8; page 245, number 3; page 247, numbers 1-5; page 248, numbers 1-5; page 251, numbers 1-2, and page 253, number 3

See teacher for check materials and/or post test for equivalent fractions.

C. Option Quest

1. Sobel, Maletsky Book I (Ginn) pages 267-271
2. STM I (Scott-Foresmen) pages 308-322

Each student's successes with these objectives should be carefully checked before proceeding to the next task. Degrees of success need to be adjusted to the needs of each student. A simple package like the above can be worked to fit into any level from intermediate grades to senior high school. Test references can easily be adjusted and self and post tests lifted from previously prepared tests of your own or another teacher. This sharing of skills and work seems to be a problem at our school. We have recently begun a weekly informal "sharing package" that has brought some results but not nearly enough. But nothing has fostered the exchange of expertise as much as membership and active participation in professional organizations such as NCTM math groups. The people you meet and the materials you share cannot be matched.

We have discussed the gradual transition of a traditional, teacher center, lecture oriented classroom into an individualized, student centered, activity oriented setting. You must want to help each child to his best to make the procedures described here work.

It has been my experience that when youngsters see teachers caring enough to give individual attention as a matter of policy many of the time consuming jobs of discipline and motivation can be minimized. Time saved here can be spent on more individualization and the whole thing snowballs very quickly. I am convinced that no special talent or personality is required for this type of classroom. Any teacher by using the techniques mentioned can show students by his actions that he cares. Once the students see this evidence the sky is the limit.

I have made the transition; I have seen many others do likewise. Will you begin? Today?

BIBLIOGRAPHY

"Instructional Systems for Students with Learning Disabilities" CEMREL INC., 10646 St. Charles Rock Road, St. Ann, Missouri 63074.

"A Comprehensive Remedial and Developmental Program for Disabled Learners at the Junior High School Level: William Page, Program Director.

Don A. Buckeye et al: *A Cloudburst of Math Law Experiments,* Midwest Publications Inc., Birmingham, Michigan, 1971

Robert E. Eicholz et al: *School Mathematics I, II Duplicator Master,* Addison-Wesley Publishing Co., Menlo Park, California, 1971

All Six Books by Martin Gardner (See Books in Print at your library)

Robert S. Graflund, *Patterns and Puzzles in Modern Math,* J. Weston Walch, Portland, Maine

Impact of New Ideas in Education, Vol. IV, No. 2, School District of University City, Missouri 63130, 1967

Hally F. Irving, George Eastman, *Highway to Math Fun,* Harr Wagner Publishing Company, San Francisco, California

Harold R. Jacobs, *Mathematics a Human Endeavor,* W. H. Feens and Co., San Francisco, California

Johnson, Honset et al: *Activities in Math First and Second Course,* Scott Foresman, Chicago, Illinois

Oscar Schaaf, Scott McFadden, *Synchro Math Experiences,* Lyons & Carnahan, Chicago, Illinois, 1972

Sobel, Maletsky, *Developing Mathematical Ideas Book I,* Ginn & Co., Boston, Massachusetts, 1966

Von, Ebrgen, Hartang et al: *Seeing Through Mathematics Book I,* Scott-Foresman Co., Chicago, Illinois, 1962

HOW MANY ARE THERE?

by E. Sherman Grable

University of Richmond, Virginia

One of the most fascinating chapters in the history of mathematics is that which deals with the counting process. From beginnings shrouded in the haze of antiquity, the story unfolds as a long struggle to establish the infinite (more specifically, infinite sets and certain infinite processes) as a tractable mathematical entity. This objective was finally achieved in the brilliant studies of the German mathematician Georg Cantor (1845-1918). And only recently (from 1938 to 1963), the efforts of Kurt Gödel and Paul Cohen have provided the answer to an important question left unanswered by Cantor, and a whole new area of investigation known as non-Cantorian set theory has been opened up.

Counting is a process that almost everyone understands, at least if it isn't carried too far. The fundamental concept of a one-to-one correspondence seems to be almost as old as the proverbial hills. It is seen in the shepherd adding one pebble per sheep to a growing pile as he attempts to measure the size of his flock; in the notches on the tally sticks used in keeping the official accounts of the English Court of Exchequer until 1826; and even in the little child carefully checking off on his fingers as he proudly counts from one to five. Long before the development of number words, one-to-one correspondences served mankind in attempts, however crude, to answer the question, "How many are there?"

Just how far the counting of the elements of a set extended, however, has varied widely. Tobias Dantzig, in his classic *Number, The Language of Science,* [1]* points out that the Bushmen of South Africa have number words only for *one, two,* and *many,* indicating an extremely limited number sense and a correspondingly limited extension of this sense in distinguishing the sizes (i.e., how many members) of different collections. The essential abstractness of the counting process has also presented difficulties. Dantzig reports an Indian tribe in British Columbia with several sets of number words: one for counting flat objects and animals, another for round objects and time, another for canoes, another for measures, and finally one for counting when no definite objects are referred to. Apparently the cardinal sameness of a set of five men and a set of five canoes was not realized until long after the specialized counting words were well established. Certainly the concept of cardinal number of a set that we sometimes use today as being the set of all sets equivalent to the given set represents a degree abstraction and mathematical sophistication that many people find difficult to comprehend.

How many students are enrolled in a certain mathematics class? How many prime numbers are less than 100? How many grains of wheat made up the entire wheat crop harvested throughout the world during the calendar year 1973? For the first question, you can provide the answer readily for any of your own classes. We would all arrive at the same answer—25—for the second question. As to the third, certainly no one will ever know the exact answer but—and this is important—the answer exists and is a fairly large but certainly finite number.

But what about infinite sets? In what sense can the elements of an infinite set be counted? Are there more rational numbers than integers? Are there more points inside a square than on one side of the square? Prior to Cantor, only tentative

*Numbers in square brackets refer to the references listed at the end of this article.

and abortive efforts had been made to answer such questions. For example, Galileo (in *Dialogs Concerning the New Sciences*, 1636) pointed out (using modern terminology) that the set of positive integers that are perfect squares, [1, 4, 9, . . .], is a proper subset of the set of all positive integers but yet each positive integer is a square root of exactly one integer in the subset of squares; i.e., that the correspondence $n \leftrightarrow n^2$ establishes a one-to-one correspondence between the set of all positive integers and the subset of those integers that are perfect squares. He then in effect throws up his hands in dismay and concludes that characteristics such as "equal," "less than," and "greater than" cannot be used in connection with infinite sets.

The specific characteristic that Galileo observed, namely that an infinite set can be placed in one-to-one correspondence with a proper subset of itself, has come to be known as Galileo's paradox. It is often used as the definition of an infinite set. But for almost 250 years following Galileo, the infinite was viewed quite warily as being beyond the bounds of mathematics. One could talk about such things as, for example, that the value of $1/n^2$ approached zero as n increased without bound, but one should not attempt to treat the infinite as an achievable entity.

As a basis for understanding Cantor's remarkable achievements, some agreements on definitions and terminology are necessary.

DEFINITION 1. Let n be any positive integer. A set S is said to have *cardinal number* n if and only if there exists a one-to-one correspondence between the elements of S and the set of integers [1, 2, 3, . . ., n].

The setting up of this correspondence is the usual manifestation of the counting process. We say one-two-three- and so on, accounting for one element of the set at a time, until we reach the positive integer n that corresponds to the final element of the set. Then we say that S has n elements, or that the cardinal number of S is n. It is easy to prove that a set cannot have two distinct cardinal numbers. Most authorities prefer not to assign a cardinal number to the null set.

DEFINITION 2. A non-empty set S is said to be *finite* if and only if its cardinal number is a positive integer.

DEFINITION 3. A non-empty set that is not finite is said to be *infinite*.

If A and B are *finite* sets with distinct cardinal numbers, any attempt to pair off the elements of A with those of B will be unsuccessful, since there necessarily will be unmatched elements in the set with the greater cardinal number. Cantor reasoned that this criterion could be extended to *infinite* sets, and invented a new kind of number, the so-called transfinite numbers, to serve as cardinals for infinite sets. This gave a basis for extending the concept of *equivalence* to infinite sets.

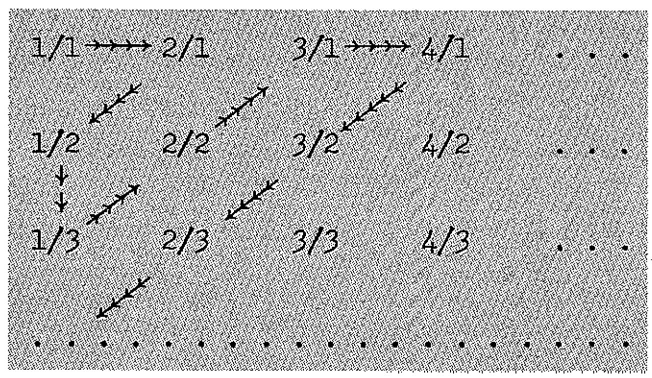
DEFINITION 4. Two sets, finite or infinite, are *equivalent* if and only if their elements may be placed in one-to-one corretspondence.

DEFINITION 5. An infinite set is said to be *countable*, or *denumerable*, if and only if it is equivalent to be set of positive integers.

DEFINITION 6. The cardinal number of a countable set is α, the first (i.e., smallest) transfinite cardinal number. (Cantor used the Hebrew *Aleph-nought* as the symbol for this first transfinite number.)

Now, countable sets certainly exist. For example, the set of all positive integers is countable since every set is equivalent to itself. Indeed, any set whose elements can be arranged in an infinite sequence $a_1, a_2, a_3, \ldots$, is countable, since the sequencing process amounts to setting up the necessary correspondence with the set of positive integers. Using the sequence $a_1 = 0$, $a_2 = 1$, $a_3 = -1$, $a_4 = 2$, $a_5 = -2, \ldots$, it is seen that the set of *all* integers is countable. Galileo's paradox is now resolved—the subset of integers that are perfect squares is indeed equivalent to the set of all positive integers, so there *are* just as many of the perfect squares as there are positive integers.

How about the set of rational numbers? Surely the rationals are more numerous than the integers, since the rationals form a *dense* set—i.e., there is a rational number (actually, infinitely many) between any two distinct rationals. But no! Arrange the positive rationals in any array as follows:



Extended sufficiently far, every rational number must eventually appear in this array—specifically, the rational p/q will appear at the intersection of column p and row q. The desired one-to-one correspondence with the set of positive integers can be realized by following the path suggested by the arrows. Thus $a_1 = 1/1$, $a_2 = 2/1$, $a_3 = 1/2$, $a_4 = 1/3$, $a_5 = 3/1$ (omit 2/2 since it has already been counted as 1/1), $a_6 = 4/1, \ldots$, whence the positive rationals are countable. As illustrated above with the set of all integers, it is now easy to see that the set of all rationals is countable, so there are just as many rational numbers as there are integers.

If an array similar to the one above but with the elements of any countable set A_1 forming the first row, those of a countable set A_2 the second row, etc., the same pattern of reasoning yields the following result:

THEOREM 1. The union of a countable collection of countable sets is a countable set.

An algebraic number is a real number that can be a root of an equation of degree n (n a positive integer) with integral coefficients. For example, $\sqrt{2}$ (a root of $x^2 - 2 = 0$) and $\sqrt[3]{2 + \sqrt{3}}$ (a root of $x^6 - 4x^3 + 1 = 0$) are algebraic numbers. By an ingenious argument, Cantor showed that it was possible to arrange all such equations in a sequence, so that the set of equations is countable. Then, since each of the equations has only finitely many algebraic numbers as roots, the set of all algebric numbers is countable.

It might seem that when the set of rational numbers is extended to include all the irrational algebraic numbers, the resulting set (i.e., the set of all algebraic numbers) would include most of the set of real numbers. But it turns out that transcendental numbers (those real numbers that are not algebraic) are by far the most numerous of all the real numbers and that in fact the set of real numbers is *not* countable. Cantor's proof of this fact is a model of simplicity. Assume that the converse is true — i.e., that the set of real numbers is countable. Then it must be possible to arrange the set in some sequential order. Applying what has come to be known as Cantor's "diagonal" argument, it is easy to show [2] that any such sequential listing must be incomplete — i.e., that there are real numbers that would never appear in the listing. Hence the assumption of countability must be false, and the uncountability of the reals follows. The symbol "c" (from continuum) is usually used to designate the cardinal number of the reals.

Given a non-empty set A, the power set of A is the set of all subsets of A. Cantor established that there are infinitely many transfinite numbers when he proved

THEOREM 2. A non-empty set A cannot be equivalent to its power set.

If the cardinal number of a finite set is n, then the cardinal number of its power set is 2^n. Extending this notation to transfinite cardinals, the cardinal number of the power set of the set of integers is indicated by 2^{α}. It can be shown [2] that the set of reals and the power set of the integers are equivalent, and hence that $c = 2^{\alpha}$. The set of all points on a line segment also has cardinality c, as does the set of all points of a square, of a cube, and even of a "solid" figure in n-dimensional space. But by Theorem 2, the power set of, say, the set of points of a square has cardinality denoted by 2^c, and 2^c must be a cardinal greater than c.

Is there a cardinal between α and 2^{α} — i.e., does there exist an infinite set whose elements are more numerous than the integers but less numerous than the reals? Cantor considered this question but was not able to resolve it, although he suspected that no such set exists. Through the years the non-existence of such a set has come to be known as the "continuum hypothesis." The question was answered only in 1963 by Paul J. Cohen, of Stanford University, building upon earlier work by Kurt Gödel. Cohen's answer was surprising— it was "yes and no," and as mentioned at the beginning of this article, his work has opened up a new branch of set theory. In what may be called "Cantorian" set theory, it is often assumed that, from any given collection of sets, a new set can be formed made up of exactly one element from each set in the given collection. Called the axiom of choice, this assumption troubled many mathematicians, and was studiously avoided if possible. In 1938, Gödel proved that if set theory as developed along conventional lines (but without the axiom of choice) is consistent, then it remains consistent if the axiom of choice also is used. Further, he showed that adjoining the continuum hypothesis likewise could not *introduce* an inconsistency. To put it another way, he did not *prove* the continuum hypothesis, but rather proved that it *could not be disproved.* Thus, the axiom of choice and the continuum hypothesis are *relatively* consistent. In 1963, Cohen successfully constructed a model of "non-Cantorian" set theory in which he assumed not the axiom of choice but rather a denial of this axiom. In this new model, he was able to show that the continuum hypothesis is false, i.e., that there can be sets with cardinal number between α and 2^{α}. Thus, the "yes and no" answer to Cantor's question. The interested reader is referred to the excellent article by Cohen and Reuben Hirsch [3] for a fascinating account of these dramatic developments.

References

1. Tobias Dantzig, *Number, The Language of Science,* Fourth edition, The Macmillan Company, New York, 1954.
2. Garrett Birkhoff and Saunders MacLane, *A Survey of Modern Algebra,* Third edition, The Macmillan Company, New York, 1965.
3. Paul J. Cohen and Reuben Hirsch, *Non-Cantorian Set Theory,* Scientific American, 217:15, December, 1967.
4. Paul J. Cohen, *Set Theory and the Continuum Hypothesis,* W. A. Benjamin, Inc., Menlo Park, 1966.

A Good Mathematics Teacher Is . . .

By ROBERT HAMADA
President, California Mathematics Council

A good mathematics teacher is like:
A FORD: *He has better ideas.*
COKE: *He's the real thing.*
PEPSI: *He's got a lot to give.*
PAN-AM: *He makes the going great.*
CONTINENTAL: *He has pride.*
BAYER ASPIRIN: *He works wonders.*
GENERAL ELECTRIC: *He lights your path.*
HALLMARK: *He cares enough to give the very best.*

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