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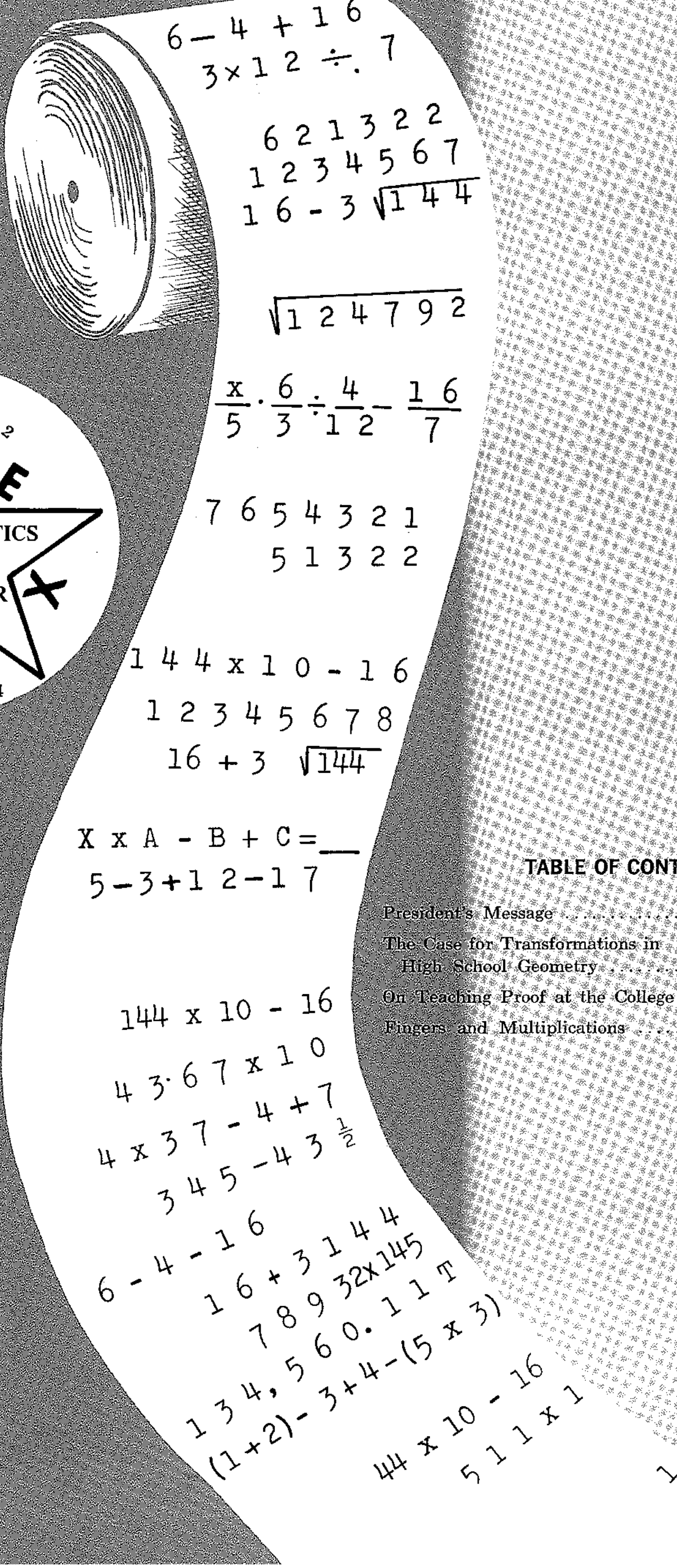
TEXAS MATHEMATICS TEACHER

March 1974



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TEXAS MATHEMATICS TEACHER is the official journal of the Texas Council of Teachers of Mathematics. The views expressed are the contributor's own and are not necessarily those of the publisher or the editor. All manuscripts and correspondence about this publication should be addressed to Mr. J. William Brown, *Texas Mathematics Teacher,* 100 So. Glasgow Drive, Dallas, Texas 75214.

TEXAS MATHEMATICS TEACHER is published quarterly by the Texas Council of Teachers of Mathematics. Payment of membership fee of $3.00 entitles members to all regular Council Publications.

President's Message

There has been much conversation lately about "doing away with 'new' math." Is it "new" math? What we are really interested in is the "new approach" to mathematics.

Emphasis is placed on understanding as opposed to memorizing a rule and working hundreds of problems involving the rule. Memorization has its place in the new approach and follows logically as the concept is internalized. For example, there is a visualization of six sets of eight objects when 6 x 8 is approached.

The four step process — concrete, semi-concrete, semi-abstract, and abstract — is utilized in presenting concepts. The student is advanced from one step to another as quickly as he indicates he is ready. As mathematics teachers, we must diagnose student needs and guide the student as he progresses through a meaningful program.

A real "plus" for the new approach is its motivating force. When concepts are internalized, practice through meaningful activities and games gets the job done and in a way that has caused many students to verbalize, "I like math," "math is fun," or "math time again, yea!"

Our opportunity today is to look critically at what we are doing and determine its value in relation to accomplishing our goal of meaningful mathematics for each student. In so doing, we will perhaps alter some approaches, but we will continue to present concepts in the "new approach" way. We know its value. Let's provide students the best!

The annual meeting of the National Council of Teachers of Mathematics is in Atlantic City April 17-20. It will be a great meeting! I hope you are planning to attend. I will be serving as your delegate to the national math assembly on the 16th. Please let me hear from you if you have resolutions to submit.

Spring is the time for many local councils to be conducting math meetings. Watch for information about workshops and meetings in your area. Attend! You'll be glad you did. Your council is encouraged to provide sessions for teachers at all levels. Call on Texas Council to help you.

CORRECTION

In the January, 1974, issue of *Texas Mathematics Teacher*, in the article entitled "A Payments Paradox," page 13, line 11 of column 1 should have read "Job B" instead of "Job A."

ATTENTION:

NUMBER ONE ON THE METRIC HIT PARADE

What can a music teacher do when his school decides to "go metric" for a month? This was the question confronting Charles Rinehart at the Campus School in Oswego, New York. His response was to use rhyme and rhythm to teach the new metric vocabulary. His song, "Halve Your Meter," has become an instant hit due to its catchy melody and painless introduction to the metric system. From the standpoint of a music teacher looking at the metric system, Charles says, "How pleasant it is to have a spoonful of sugar to help the metric go down."

The words and music of the tune appear in the May, 1974, issue of SCHOOL SCIENCE AND MATHEMATICS. If you would like a reprint of the article, which includes the words and music to the tune, simply send a stamped, self-addressed envelope to:

School Science and Mathematics Association
P. O. Box 1614
Indiana University of Pennsylvania
Indiana, Pennsylvania 15701

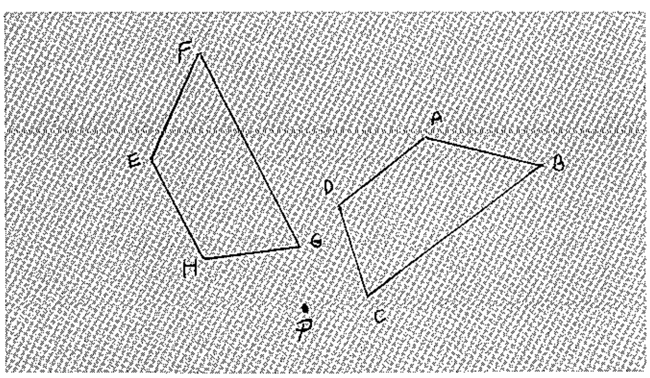
THE CASE FOR TRANSFORMATIONS IN HIGH SCHOOL GEOMETRY[1]

by Zalman Usiskin
University of Chicago

Five years ago, few teachers had heard about transformations except in college linear algebra courses, and certainly not too many teachers were thinking about them in high school geometry, even though there were available materials.[2] Yet today, in an increasing number of texts at elementary, junior high, and high school levels, for both slow and fast learners, transformations are mentioned, given lip-service, given sections, or studied in detail. I have been one of those people who have encouraged the use of transformations both in algebra and in geometry. The purpose of this article is to give one view of the case for transformations. After a brief introduction to transformations, 5 reasons are given for their inclusion in school curricula.

What are transformations?

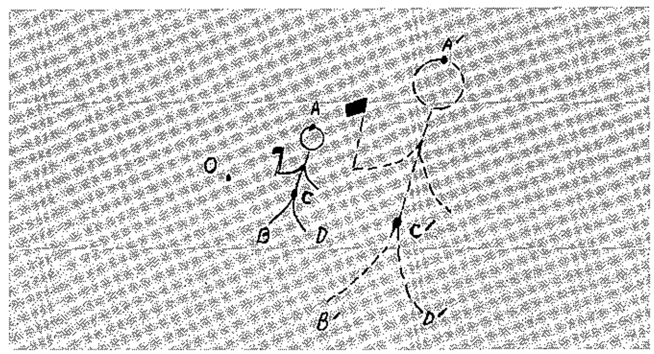
A transformation is a correspondence (often restricted to be 1-1) between the points of one set and the points of a second set. Below is pictured one example of such a correspondence. Point A corresponds to E, B corresponds to F, C corresponds to G, and D corresponds to H. All other points of ABCD correspond to points of EFGH. What do we call this correspondence? It looks like we could have turned ABCD 80° around the point P (stick a needle into P and physically turn the page to verify) and it would fall upon EFGH. So we call this correspondence a *rotation*. In fact, this particular rotation has center P and magnitude 80°. We call EFGH the *image* of ABCD under this rotation. ABCD is the *preimage*. We say that the rotation *maps* ABCD to EFGH.



In the physical model, all points — not just those on ABCD — have images. And so we speak of transformations of the plane even though we may focus only on a few points and their images.

Notice that the distance between two preimages (B and D, for example) is equal to the distance between their images (F and H). This is a characteristic of all rotations — they *preserve* distance. Not all transformations preserve distance. Below is one which does not — a *size transformation* (also called a dilatation).

In the size transformation, a center and magnitude are chosen. Here we have picked O as the center and 2½ as magnitude. For each point A on the preimage, we find image A′ by choosing the point on OA which is 2½ times as far from O as A is. That is, OA′ = 2½ · OA. This process is repeated with as many points as are needed to determine the image.



The preimage and image probably look similar to the reader. However, the standard definition of "similar" does not cover such complicated figures. Yet many applications of similarity (patterns, scale models and drawings, photographs, magnifying figures) require a general notion of similarity. This gives the first major building block for the case for transformations.

1. Transformations enable one to deal with a much greater variety of figures in geometry.

Here are possible definitions of congruence and similarity as seen from a transformation point of view. Notice that they apply to all figures, not merely segments or angles or triangles.

Definition: Two figures α and β are *congruent* if and only if there is a distance-preserving transformation which maps α onto β.

Definition: Two figures α and β are *similar* if and only if there is a distance-multiplying transformation which maps α onto β.

By "distance-multiplying" we mean that if A and B are any two preimages and A′ and B′ their images, then $\frac{A'B'}{AB}$ is a constant. This constant is the ratio of similitude. In the above example, this constant ratio is 2½. That is, the size transformation in some sense multiplies distances by 2½.

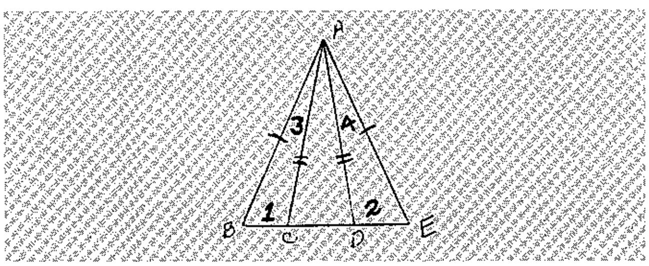
With these definitions, it is *natural* to ask: "When are two triangles congruent?" "When are two polygons similar?" From these questions, the standard content of geometry can be developed. Under traditional courses, it is doubtful that students think of congruence or similarity as being in any way natural. This is a second major joint.

2. Transformations bring geometry much closer to the intuition of the child.

The early geometry experiences of the child in grades K-8 lead the child to understand that congruence and similarity depend upon size and shape. From this intuition, the child ought to realize that assembly-line production depends upon the building of congruent parts, that xerox duplication gives us congruent pages, that he or she is congruent to his image in a mirror (at a given point in time), that what you see through a telescope or microscope is similar to what is behind the lens, and so on.

However, the standard high school course in geometry divorces these concepts from the intuition of the child by (1) ignoring the very existence of the world it was supposed to model and (2) setting up different definitions of congruence and similarity for each type of figure even though the underlying intuition is the same.

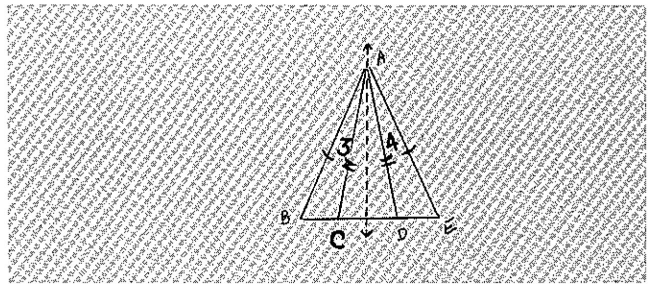
This idea can be illustrated in another way. Below is a figure found in geometry texts. Some information is given, and let us suppose that the child is asked to deduce that $\angle 3 \cong \angle 4$.



How does the child *know* that $\angle 3 \cong \angle 4$. Certainly not because he proves it. The proof does not convince most children. He knows that $\angle 3 \cong \angle 4$ because in some way the figure is *balanced.* For this reason, the child would balk at being asked to prove $\angle 2 \cong \angle 3$. The figures in geometry texts are almost always symmetric. Symmetry is another important concept which is most easily defined in terms of transformations.

Definition: A figure is *symmetric* if there is a non-trivial* distance-preserving transformation which maps the figure onto itself.

(*The transformation which maps each point onto itself is distance-preserving. (Think of a rotation of 0°.) If this transformation were not excluded from the definition as being trivial, then every figure would be symmetric.) The figure used above is reflection-symmetric. There is a reflection over the disector of $\angle$ CAD which maps the figure onto itself. Notice that this reflection maps $\angle 3$ onto $\angle 4$. For this reason, they are congruent.



Because transformations bring geometry closer to the child's experience, they can make the development easier. (This is not the case with every development using transformations; some writers insist on making simple mathematics as difficult to learn as they can.) Furthermore, the geometry can be made much more visual and physical. This makes the use of transformations particularly appropriate for students who do not easily grasp proof concepts. They can at least understand the material about which they are being asked to make deductions. This is the third major reason for transformations.

3. Transformations make geometry more accessible to the slower student.

It is notable that UICSM decided to use transformations almost exclusively in designing a course in geometry for slow 8th graders.[3] It is also notable that here is a case of "new math" being easier — it is taught in elementary schools in Europe — and this goes against the experiences of many of the new curricular materials of the 60's, so that teachers don't believe it until they teach transformations.

Mathematics does not have to be hard to be good.

The generality with which transformations can be used in geometry makes them much more applicable in later courses. Put another way, there aren't too many triangles in the second-year algebra course, so the standard geometry course cannot easily be applied here. Here are just a few of the applications of transformations which are possible in later high school courses.[4]

a. The congruence of parabolas $y = x^2$ and $y = (x - h)^2 + k$ and $y = -x^2$, for example.
b. Reflections over the line $y = x$ for inverse functions.
c. Symmetry of the conics.
d. Congruence of the graphs of $y = \sin x$ and $y = \cos x$, application to phase shift.
e. The similarity of all parabolas (take a magnifying glass to a thin parabola and it looks wider).
f. Rotating 90° to get slopes of perpendicular lines.
g. Polar coordinates as arising out of a rotation and size transformation.

h. Connections with real number and complex number addition and multiplication.
i. Periodicity of functions as an instance of translation symmetry.

Other applications are related either to matrices or to groups, which arise naturally if one has transformations.

These connections with what used to be exclusively college-level mathematics has forced *every* curriculum project in the last five years for good students to introduce transformations early in their development.[5]

4. Transformations provide assistance (and a strong weapon) for future work in mathematics.

It is not geometry which has no applications in later mathematics, it is the *standard course* in geometry which has so few applications.

Indeed, transformations are examples of functions and hence useful in preparing to study analysis. Knowledge of transformations is necessary in studying groups. The intimate association between transformations and matrices is fundamental in linear algebra. And every geometer must know about transformations because some theorems have no other simple proofs — except those which use transformations.

There are other concepts which are useful in later mathematics, and coordinates and vectors have been posed as possible tools with which to approach geometry. Thus far, all developments using these ideas available in the U.S. seem to require too many algebra skills for the average student. (The traditional unifying concept, proof, is, of course, not geometric but a logical concept.) And these methods do not enable one to easily approach the fundamental concepts of congruence, symmetry, and similarity and use these ideas to deduce properties of figures. We thus come to the last reason given here for using transformations in geometry.

5. Transformations give a unifying concept to the geometry course which is geometric in nature.

The reader may now wonder if there are any arguments against the use of transformations. The only one I have heard is that teachers are not ready to teach transformations. This has not been a great problem in my geographic area nor in some other areas of the United States. If material is really at the level of the average student, then it can be understood by teachers with no special training. Experience has shown that teachers go through materials more slowly in their first year of teaching transformations than in their second, and this would be expected with any new materials. In short, there is no case against the use of transformations in geometry — their time has come and they are here to stay.

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[1]Adapted from two talks given at the CAMT, Austin, Texas, November 2, 1973.

[2]Z. Usiskin, "Transformations in High School Geometry Before 1970," to appear in *The Mathematics Teacher* sometime in 1974.

[3]UICSM, *Motion Geometry*, Harper and Row, 1969.

[4]All of these applications can be found in some experimental 11th grade materials, *Intermediate Mathematics*, available from this writer.

[5]For example, the Secondary School Mathematics Curriculum Improvement Study (SSMCIS), centered at Columbia University, and the Comprehensive School Mathematics Project (CSMP), centered in Carbondale, Illinois.

ON TEACHING PROOF AT THE COLLEGE LEVEL

by Marvin L. Bittinger
Indiana University-Purdue University at Indianapolis

Consider the following two learning theories regarding the teaching of proof at the college level:

Theory 1. Study axiomatic systems, and let the proof strategies evolve from this study.

Theory 2. Study proof strategies first. Then use them to more effectively study and do proofs related to axiomatic systems.

Which is best? On thinking about this, one quickly realizes that the arguments are analogous to "Which comes first, the chicken or the egg?" Theory 1 has probably been used most at the college level. The argument presented in the rest of the paper will be for the use of Theory 2 and how this can be accomplished.

A study by Morgan [3] provides strong arguments against Theory 1, under the assumption that, in fact, Theory 1 is what is used most in teaching proof at the college level. Morgan found that of students who had 30+ hours of mathematics at the college level only 80% knew the starting assumption for a direct proof [To prove $P \rightarrow Q$, assume P (and deduce Q)]; only 37% knew the starting assumption for a proof by contrapositive [To prove $P \rightarrow Q$, prove $\sim Q \rightarrow \sim P$, by assuming $\sim Q$ (and proving $\sim P$)]; and only 33% knew the starting assumption for a proof by contradiction [To prove $P \rightarrow Q$, assume its negation $P \wedge \sim Q$, (and derive a contradiction $R \wedge \sim R$)]. *No wonder students cannot create proofs by contradiction — they do not even know how to get started!* The findings of this study raise serious questions regarding the preparation of an under-graduate mathematics major to do graduate work. Questions also arise about the preparation of secondary mathematics teachers regarding the teaching of proof.

It is my conclusion that we ought to be using Theory 2 and the remainder of the article is devoted to how to carry this out. Theory 2, in effect, infers that a certain learning set be implanted in the minds of students. Each time a student encounters a sentence to be proved he considers a *set* sequence of steps. The learning set (or strategy) in analogous to a widely accepted strategy for solving applied problems in algebra.

Algebra Problem: The length of a rectangle is 3 ft. more than the width and the area is 54 ft^2. Find its dimensions.

STRATEGY:

(1) *Translate to Mathematical Language:* In this example, this means, translate to an equation:

$$w(w + 3) = 54, \text{ or}$$
$$w^2 + 3w - 54 = 0$$

(2) *Examine the Equation, Select a Method of Solution From:*

a. factoring
b. completing the square
c. using the quadratic formula

This strategy has two clear implications in regards to the skills students should have obtained prior to a study of solutions of applied problems. (1) They need to practice the skill of translating problems to equations. This can be accomplished most effectively if done with no thought at the time of actually solving the problem. (2) Students must have also obtained the skill at using any of the methods of solving the translated equation in this case of solving quadratic equations.

An analogous example yields the learning strategy (Theory 2) for creating proofs.

Proof Sentence: The square of every even integer is an even integer.

STRATEGY:

(1) *Translate to Mathematical Language:* This can be interpreted to mean, translate to logical symbolism:

$\forall x$, x is even $\rightarrow x^2$ is even (Quantifier could be written)

(2) *Examine the Translated Sentence, Select a Mode of Proof From:*

a. direct proof
b. proof by contrapositive
c. proof by contradiction

As with the applied problem strategy, this proof strategy has two clear implications in regards to the skills students should have obtained prior to attempting proofs of sentences. (1) Students need practice at translating sentences to logical symbolism. Often, as in the case of a conditional sentence, there are many ways of expressing a sentence in written manner, but which have the same translation to logical symbolism. For example, all of the following have the same meaning:

$\forall$ x, x is even $\rightarrow x^2$ is even
The square of every even integer is even.
x is even implies x^2 is even
x is even only if x^2 is even
x is even is a sufficient condition for x^2 to be even
x^2 is even is a necessary condition for x to be even

(2) Students must have practice at using the possible modes of proof for a given sentence. Clearly, even if students have the previous skills it still does guarantee that they will be able to create a proof of a given sentence, but at least they are much closer to success than if they did not practice this strategy.

For the remainder of this article we will expand upon this strategy with various types of sentences.

A. *Conditional.* The following are several ways in which conditional sentences are expressed followed by the way they are expressed in logical symbolism.

If P, then Q
P implies Q
Q if P
P only if Q
P is a sufficient condition for Q
Q is a necessary condition for P

$P \rightarrow Q$

The following are possible modes of proof for conditionals. (1) *Rule of Conditional Proof.* Assume P, deduce Q. The motivation for this mode of proof follows from the truth table for $P \rightarrow Q$.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

When the antecedent P is false the conditional $P \rightarrow Q$ is true from the truth table. So, the only case to check is when P is true. Making P an axiom temporarily and using all previous axioms and theorems we try to deduce Q. For example,

Prove: x is even $\rightarrow x^2$ is even.

Proof. Assume x is even. Then $x = 2k$, for some integer k.

Then $x^2 = (2k)^2 = 2(2k^2)$, so x^2 is even.

Such seemingly trivial examples help the student to practice the proposed proof strategy in a familiar axiomatic system, so the proof strategy becomes the new learning, rather than the axiomatic system.

(2) *Proof by Contrapositive.* Using truth tables one can verify that a conditional sentence $P \rightarrow Q$ is equivalent to its contrapositive $\sim Q \rightarrow \sim P$. This yields another mode of proof for $P \rightarrow Q$:

Assume $\sim Q$, deduce $\sim P$. For example, *Prove:* x^2 is even $\rightarrow$ x is even

Proof. Write the contrapositive: x is odd $\rightarrow x^2$ is odd

Assume x is odd. Then $x = 2k + 1$, for some integer k.

Then $x^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1$, so x^2 is odd.

When a new equation solving technique is introduced students need to practice it. Similarly, when a new proof technique is introduced students need to practice it. Sentences which lend themselves to proof by contrapositive should be used for such practice. The following are some further examples:

a. x^2 is odd $\rightarrow$ x is odd
b. If two lines are cut by a transversal so that alternate interior angles are congruent, then the lines are parallel.

(3) *Proof by Contradiction.* This is a mode of proof for any sentence, in particular for conditionals. We will treat this in detail later.

B. *Biconditional.* The biconditional sentence $P \rightarrow Q$ occurs in the following forms.

P is equivalent to Q
P if and only if Q
P iff Q
P is a necessary and sufficient condition for Q

The following are possible modes of proof for biconditionals. (1) Prove $(P \rightarrow Q)$ & $(Q \rightarrow P)$. This is actually the definition of a biconditional sentence. There are two parts to such a proof.

(2) Prove $(P \rightarrow Q)$ & $(\sim P \rightarrow \sim Q)$. This mode of proof is accomplished by proving one conditional and the contrapositive of the other.

For example, to prove

x is even $\rightarrow x^2$ is even

one might prove

a. x is even $\rightarrow x^2$ is even
b. x is odd $\rightarrow x^2$ is odd

Again, trivial examples allow greater emphasis on the proof strategy.

(3) *Iff-String.* This mode of proof of a conditional sentence $P \rightarrow Q$ is illustrated below.

$$P \leftrightarrow S_1$$
$$S_1 \leftrightarrow S_2$$
$$\vdots$$
$$S_n \leftrightarrow Q$$

That is, to prove $P \rightarrow Q$ we produce a string of equivalent sentences leading from P to Q. This could also be accomplished in another way:

$$P \rightarrow Q$$
$$Q \rightarrow S_1$$
$$\vdots$$
$$S_n \rightarrow P$$

That is, prove $P \rightarrow Q$, $Q \rightarrow S_1$, ... $S_n \rightarrow Q$, a string of conditionals producing a cyclic argument from P to Q and back to P.

C. *Universally Quantified Sentences.* The universally quantified sentence

For every x, P(x), or $\forall x$, P(x)

is proved by showing that for every x in a specified universal set, that P(x) is true. For finite sets this is just a matter of substitution, but for infinite sets proofs usually rely on universally quantified axioms. For example,

Prove: $\forall x, 1 < x \rightarrow 1 < x^2$

Proof. Let x be fixed but arbitrary. Assume $1 < x$. Then $x > 0$, so $1 \cdot x < x \cdot x$, or $x < x^2$.

A possible reaction to this is "You only proved it for one x, not for all." One needs to point out that if it were possible to repeat the proof for every x in the universal set each proof would be the same.

D. *Existentially Quantified Sentences.* The existentially quantified sentence

There exists an x such that P(x), or $\exists x$, P(x)

is proved by showing that there exists an x in a specified universal set for which P(x) is true. Some examples are quite trivial, such as

$\exists x$, $x = 0$, here there is only one x
$\exists x$, $\cos x = 1$, here there are many x's

Using proof by contradiction one can actually prove the existence of an object possessing a certain property without directly displaying that object.

E. *Proof by Cases.* To prove a sentence of the type

(P or Q) $\rightarrow$ S

it can be shown by truth tables that it is sufficient to prove

$(P \rightarrow S)$ & $(Q \rightarrow S)$

An example might to to prove ($a = 0$ or $b = 0$) $\rightarrow ab = 0$.

Situations can arise where a sentence $P \rightarrow Q$, lacking any word "or" can be proved if first proves an intermediary "or" sentence. For example,

Prove: x is an integer $\rightarrow x^2 + x$ is even.

Proof. First of all, x is an integer $\rightarrow$ x is even or x is odd. Now we have an "or" sentence and we can use proof by cases.

Case 1. Prove. x is even $\rightarrow x^2 + x$ is even.

Case 2. Prove. x is odd $\rightarrow x^2 + x$ is odd.

We omit these proofs.

The real art in producing a successful proof by cases is selecting the appropriate cases, as in the following absolute value proofs:

$$|x| \geqslant 0, \quad |xy| = |x| \cdot |y|, \quad |x^2| = |x|^2$$

F. *Mathematical Induction.* One can prove

For every natural number n, P(n), or $\forall n$, P(n)

if one can

a. Basis Step. Prove P(1)
b. Induction Step. Prove. $\forall k$, $P(k) \rightarrow P(k+1)$

There are some standard motivations for this procedure which will be omitted here. There are some points to make to aid the teaching of mathematical induction. The first is to *give many kinds of examples*, not just examples using summation of series problems. The folowing are some possibilities:

a. For every n, $\sum_{j=1}^{n} j^2 = \frac{n(n+1)\ (2n+1)}{6}$

b. For every n, $|\sin nx| \leq n\ |\sin x|$

c. For every n, $(2n)! < 2^{2n}(n!)^2$

d. For every n sets, $A_1, \ldots, A_n$,

$$C(\bigcup_{j=1}^{n} A_j) = \bigcap_{j=1}^{n} CA_j,$$

where CA is the complement of A.

In this way students become aware that mathematical induction is a very universal and useful method of proof.

Another helpful point to make in regards to teaching mathematical induction is to make use of recursive definitions like the following.

Definition of exponents. For an a and any integer $k > 1, a^1 = a, a^{k+1} = a^k \cdot a$

Definition of sigma notation. For any number $a_1, \ldots, a_k, a_{k+1}$

$$(\sum_{j=1}^{1} a_j) = a_1 \quad (\sum_{j=1}^{k+1} a_j) = (\sum_{j=1}^{k} a_j) + a_{k+1}$$

Let us do a proof by mathematical induction to point out other helpful procedures. The proof is of (a) previously stated. It is helpful to first write down P(n), P(1), P(k), and P(k+1) so that one has right in front of him what has to be proved.

$$P(n): \sum_{j=1}^{n} j^2 = \frac{n(n+1)\ (2n+1)}{6}$$

$$P(1): 1^2 = \frac{1(1+1)\ (2 \cdot 1+1)}{6}$$

$$P(k): \sum_{j=1}^{k} j^2 = \frac{k(k+1)\ (2k+1)}{6}$$

$$P(k+1): \sum_{j=1}^{(k+1)} j^2 = \frac{(k+1)\ (k+2)\ (2k+3)}{6}$$

Usually the proof of the basis step is just a matter of substitution, but the induction step requires more creativity. Students are usually taught, when proving trigonometric identies to start with one side and derive the other. Such a method along with use of recursive definitions can make the proof of the induction step almost a routine matter in some cases:

$$\sum_{j=1}^{k+1} j^2 = \underbrace{(\sum_{j=1}^{k} j^2)} + (k+1)^2, \text{ recursive definition}$$

$$= \frac{k(k+1)\ (2k+1)}{6} + (k+1)^2, \quad \text{by } P(k)$$

$$= \frac{(k+1)\ (k+2)\ (2k+3)}{6}$$

G. *Proof by Contradiction.* To prove a sentence P one assumes its negation ~P, and deduces the truth and falsity of any sentence S; that is, one proves S & ~S. In particular, to prove P → Q, one assumes P&~Q. This points out another important skill the student must have; that is, he must be able to form negations of sentences.

For example,

$\sim(P \rightarrow Q) \leftrightarrow (P \& \sim Q)$

$\sim \forall x, P(x) \leftrightarrow \exists x, \sim P(x)$

$\sim \exists x, P(x) \leftrightarrow \forall x, \sim P(x)$

The omission of quantifiers can sometimes lead to fallacious proofs such as the following. Can you find the error?

Prove. If x is rational and y is irrational, then x+y is irrational.

Proof. Assume the negation: x is rational & y is irrational & x + y is rational.

Since 0 is a rational, it can be substituted for x and

x + y is rational, or
0 + y is rational, or
y is rational, which is a contradiction

Now let us give a valid proof by involving the quantifiers.

Prove. For every x and for every y, if x is rational & y is irrational, then x + y is irrational.

Proof. Assume the negation: There exists an x and there exists a y such that x is rational, y is irrational, and x + y is rational.

Then $x = \frac{a}{b}$, for some integers a and b, and $x + y = \frac{c}{d}$, for some integers c and d. Then

$$y = (x + y) - y = \frac{c}{d} - \frac{a}{b} = \frac{cb - da}{db}$$

Therefore y is rational and we have a contradiction. It was commented earlier that proof by contradiction can actually prove the existence of an object possessing a certain property without actually displaying it. The following is such a proof.

Prove: There exists an irrational number a and an irrational number b such that a^b is rational.

Proof. Assume the negation: For every irrational number a and every irrational number b, a^b is irrational. We use the fact that $\sqrt{2}$ is irrational. Then we know that $\sqrt{2}^{\sqrt{2}}$ is irrational by the negation.

Then we also know that $\left[\sqrt{2}^{\sqrt{2}}\right]^{\sqrt{2}}$ is irrational

But $\left[\sqrt{2}^{\sqrt{2}}\right]^{\sqrt{2}} = (\sqrt{2})^2 = 2$, and we have a contradiction.

Note carefully that this proof did not actually display an irrational number a and an irrational number b such that a^b is rational, though it did prove that such numbers exist.

SUMMARY

A new teaching strategy is proposed (along with some hints) for creating proofs:

(1) Translate to logical symbolism
(2) Examine the translated sentence, select a mode of proof.

Some additional hints are as follows.

(3) After a reasonable effort with one mode of proof, try another.
(4) Examine analogous proofs for hints.
(5) Use definitions and previous theorems.
(6) Realize that trial and error are very much a part of proof creativity.

REFERENCES:

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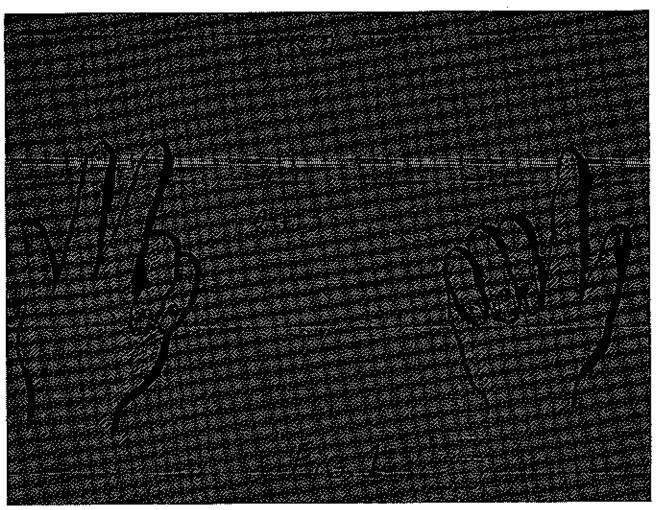
FINGERS AND MULTIPLICATIONS

by Ali R. Amir-Moéz
Texas Tech University

Have you ever used your fingers for multiplication? Here we shall explain it.

One has to know the multiplication table up to 5. Then one can use his fingers. For example,



for multiplying 7 by 8, as 7 is 2 units larger than 5, we bend 2 fingers of one hand (Fig. 1). As 8 is 3 larger than 5, we bend 3 fingers of the other hand. Thus we have 5 bent fingers and we consider 50. In one hand there are 3 fingers which haven't been bent and in the other one we have 2 unbent fingers. We multiply 3 by 2 and we get 6. Consequently, we get $7 \times 8 = 50 + 6 = 56$.

One may try other examples in order to verify the technique.

Proof: Let a and b be the two numbers. Suppose

$$a' = a - 5,\ b' = b - 5$$

and

$$a'' = 10 - a,\ b'' = 10 - b.$$

Then we observe that

$$10\ (a' + b') + a''b'' = ab.$$

DEDOS Y MULTIPLICACIONES

Por: Ali R. Amir-Moéz
Texas Tech University

¿Há empleado alguna vez los dedos para multiplicar? Ahora vamos a ver cómo hacerlo.

Se debe saber solamente la tabla de multiplicación hasta 5. Ya que se puede usar los dedos. Por ejemplo, para multiplicar 7 por 8, como 7 es 2 más grande que 5, doblamos dos dedos de una mano (Fig. 1). Como 8 es 3 más grande 5, doblamos 3 dedos de la ortra mano. Tenemos por lo tanto 5 dedos doblados y consideramos 50. Tenemos 3 dedos sin doblar en una mano y 2 dedos en la otra. Multiplicamos 3 por 2 para obtener 6. Por fin, obtenemos $7 \times 8 = 50 + 6 = 56$.

Se pueden tratar otros ejemplos para probar el método.

Verificación: Sea a y b los dos números. Supongamos

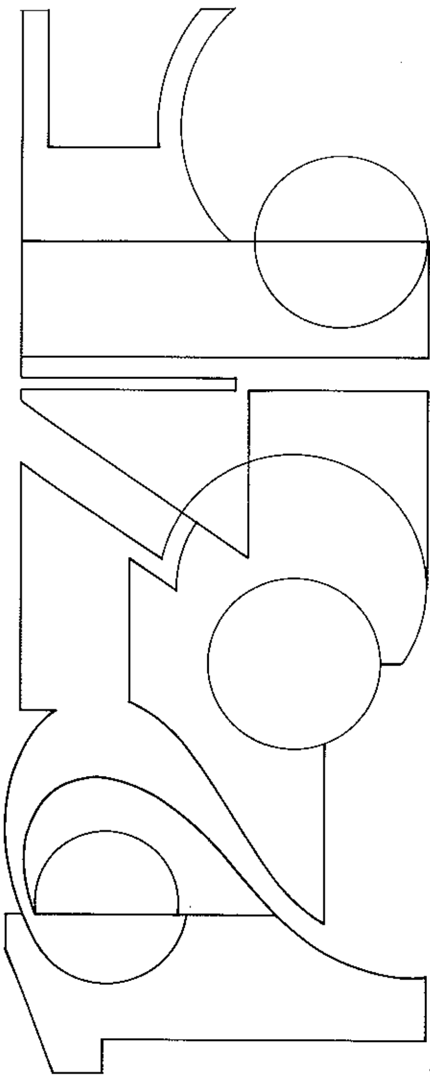
$$a' = a - 5,\ b' = b - 5$$

y

$$a'' = 10 - a,\ b'' = 10 - b.$$

Entonces observamos que

$$10\ (a' + b') + a''b'' = ab.$$



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J. William Brown, Editor
Texas Council of Teachers Of Mathematics
Woodrow Wilson High School
100 S. Glasgow Drive
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