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TEXAS MATHEMATICS TEACHER

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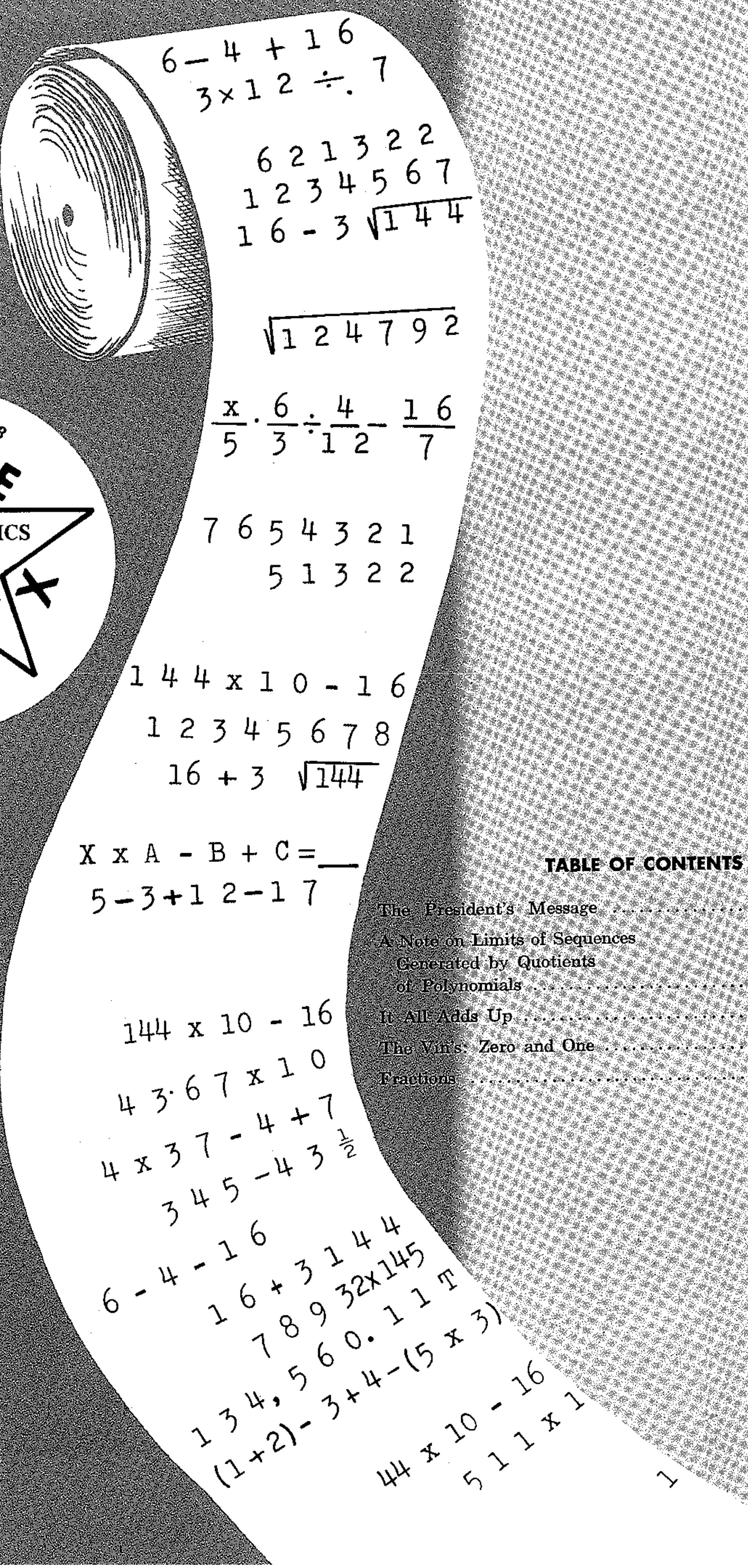


TABLE OF CONTENTS

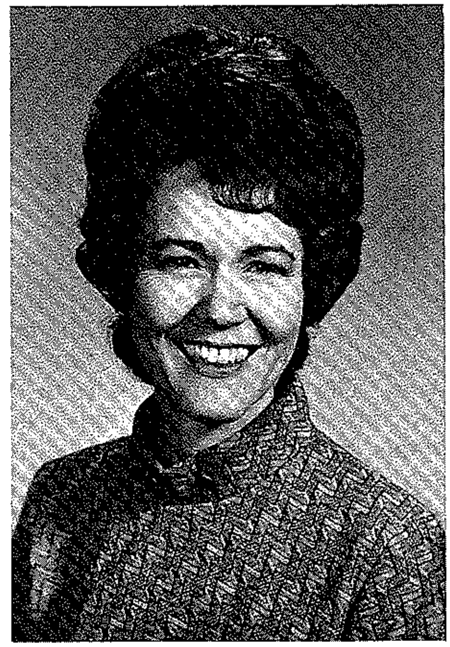
TEXAS MATHEMATICS TEACHER is the official journal of the Texas Council of Teachers of Mathematics. The views expressed are the contributor's own and are not necessarily those of the publisher or the editor. All manuscripts and correspondence about this publication should be addressed to Mr. J. William Brown, *Texas Mathematics Teacher*, 100 So. Glasgow Drive, Dallas, Texas 75214.

TEXAS MATHEMATICS TEACHER is published quarterly by the Texas Council of Teachers of Mathematics. Payment of membership fee of $3.00 entitles members to all regular Council Publications.

PRESIDENT'S MESSAGE

by

Shirley Ray



How great it is to be a part of the Texas Council of Teachers of Mathematics! This organization has made tremendous strides in recent years. The new journal provides many useful ideas through its varied and excellent articles. Many outstanding people in the field of mathematics have contributed much time to share suggestions in order to help teachers as they try to meet the needs of each of their students.

Chuck Carson has been an outstanding president and will continue to be a valuable resource person, as well as an executive board member in the role of past president. All of the officers have taken on their tasks with enthusiasm. The officers of the organization who are continuing to serve for another year are Thomas Hall and Crawford Johnson as vice-presidents, and Floyd Vest as treasurer. J. William Brown will also continue to serve as our editor. Those new to our executive board are Sister Jane Myer, who will serve as secretary; Bill Stanford, who will ge our Parliamentarian; and Madge Simon, who will be N.C.T.M. representative for the next two years. Together we pledge our support to each of you.

The Texas Council of Teachers of Mathematics is a service organization to you and its members. We urge local councils to hold workshops for teachers and will assist you in their organization and in providing speakers. Several workshops are held throughout the state annually and are among the most exciting things that have been done in the past several years. Teachers have been thrilled to spend a Saturday morning not only learning more about mathematics, but also leaving the workshop with arm loads of aids to help them in their teaching. If you are interested in holding such a meeting, work with your local council and contact us if we can be of service to you in helping to plan and co-sponsor such a workshop.

The annual meeting of the Texas Council of Teachers of Mathematics was held February 16 in Austin. The membership voted to raise the dues from $2.00 a year to $3.00 a year. As you know, the journal that you have been receiving was started over a year ago. Funds for its production have been made available through advertising. Additional funds have been provided through math workshops due to the generosity and contributions of the school districts where the workshops were held. The cost of the journal is not covered for the year 1973-74. Although we usually receive advertising from various book companies, TCTM needs to have complete assurance that the expenses of printing and distributing the journal are covered; therefore, the membership voted the increased dues. Let me urge you at this time to contribute any articles that you have which would be of interest to other mathematics teachers. We are always looking for materials that will help our teachers do a better job. Please send your items to our editor, Mr. Brown, in Dallas.

The calendar of mathematics events for the coming months is both exciting and challenging and will provide many opportunities for our own personal growth.

Held in April, of course, was the 51st annual meeting of the National Council of Teachers of Mathematics. Hope all of you were in Houston for this very fine meeting.

August 15-17 is the Fort Worth NCTM meeting. Perhaps, many of you will have opportunity to go to this as a part of your preschool inservice work. Plans are now in progress for a mathematics meeting to be held this fall either in Austin or Houston. More details will be available at a later time. So you see there are some very fine meetings ahead of us, and we look forward to seeing you there.

Conference for the Advancement of Mathematics Teaching: November 1-3, 1973, Stephen F. Austin and Driskill Hotels, Austin, Texas. Preregistration will be $5; on site, $6. Plan now to attend.

Let me urge you to correspond with me or any of the executive board if you have suggestions which will help us to have a better organization. The organization is *you;* and, if you help us, together we can strengthen the mathematics programs throughout Texas. As your executive committee, our desire is to be of service to you. I look forward to these two years as we work together and plan together for the boys and girls of Texas.

A NOTE ON LIMITS OF SEQUENCES GENERATED BY QUOTIENTS OF POLYNOMIALS

Dr. W. D. Clark
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In an attempt to evaluate the limits of sequences whose general term, a_N, is the quotient of polynomials of the symbol n, one basic technique is employed. The technique used is to divide the numerator and denominator by the largest power of n common to both and then evaluating the result using previously established basic theorems. The problem here is that sequences of this type are introduced primarily to study the basic theorems mentioned and although students readily grasp this technique this author finds that they invariably fail to realize an important by-product of this study. Problems of this type may be placed into three distinct categories. After analyzing these categories one can immediately evaluate the aforementioned limits on sight which, of course, is a valuable asset when a limit of this type needs to be calculated as a prelude to the solution of a larger problem.

Rather than give a detailed proof of the three categories they will be illustrated and the details left to the reader and/or the interested student. The three categories are:

1. The degree of the numerator being equal to the degree of the denominator.
2. The degree of the numerator being less than the degree of the denominator.
3. The degree of the numerator being greater than the degree of the denominator.

An example of a Type 1 sequence is that defined by the formula

$$a_n = \frac{4n^3 + 6n^2 + 5}{7n^3 + 16n^2 + 3n + 7}$$

which may be written

$$a_n = \frac{4 + 6/n + 5/n^3}{7 + 16/n + 3/n^2 + 7/n^3}$$

and it can be shown, by basic theorems referred to earlier, that this sequence converges to 4/7. This is merely the quotient of the leading coefficients of the polynomials. In general this is also true.

An example of Type 2 is that defined by

$$a_n = \frac{4n^3 + 6n^2 + 5}{7n^4 + 16n^3 + 2n^2 + 3}$$

$$= \frac{4 + 6/n + 5/n^3}{7n + 16 + 2/n + 3/n^2}$$

which can be shown to converge to 0. In general this also is true.

Finally, a Type 3 sequence is given by

$$a_n = \frac{4n^3 + 6n^2 + 5}{7n^2 + 2n + 4}$$

$$= \frac{4n + 6 + 5/n^2}{7 + 2/n + 4/n^2}$$

which can be shown to diverge to infinity. In general this is true when qualified by the summary below.

In summary, if a_N is the quotient of polynomials in the symbol n then;

1) If the degree of the numerator is equal to the degree of the denominator then the sequence converges to the quotient of the leading coefficients of the polynominals.

2) If the degree of the numerator is less than the degree of the denominator then the sequence converges to 0.

3) If the degree of the numerator is greater than the degree of the denominator then the sequence diverges to $\pm\infty$ depending upon the algebraic sign of the quotient of the leading coefficients of the polynomials.

These results are generally left up to the student to discover on his own but the author finds that this is one thing that just is not discovered. The reader may easily experience this failure to discover by assigning thirty problems of this type and have them turned back all worked by the technique of dividing by the largest common power of n!

The ability to evaluate the limits on sight is extremely useful when working with infinite series and in many other areas of analysis.

An interesting and rather easily obtainable extension of these ideas, which may be a good Junior Research project, is the extension to quotients of polynomials in the symbol x as x approaches $+\infty$, $-\infty$ or 0. In this same vein it is found that the extension to generalized polynominals of the ideas presented here is an interesting project for accelerated high school students.

To paraphrase John Ruskin, low-achievers need three things to be happy and successful in their work. They must be fit for it. They must not have too much of it. And they must have a sense of success in it.

Successful learning, especially for low achievers, begins with self-confidence and a sense of achievement. And that is what SUCCESS WITH MATHEMATICS brings to your classroom.

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Nothing succeeds like success.

Alexandre Dumas

Each stage is a foundation for the next. Each develops greater self-confidence. Each contributes satisfying achievement.

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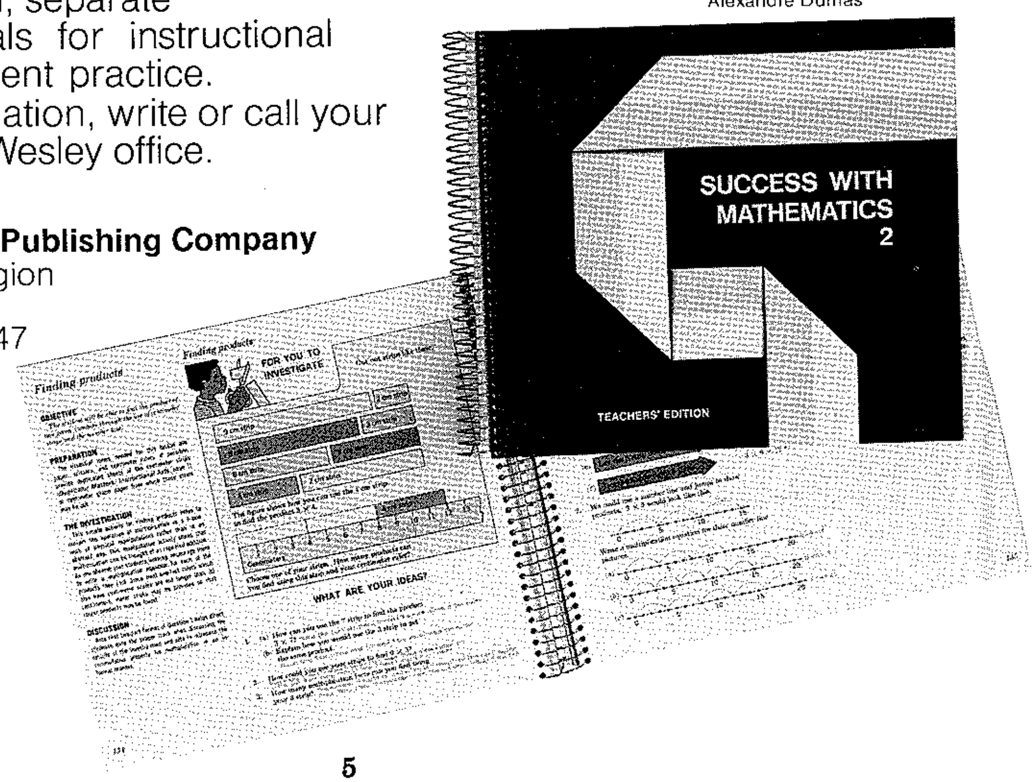

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IT ALL ADDS UP

Dr. Wallace Davis
Dean of College of Education
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The common thread that unites all of us in education is our belief in the teachers and their commitment to the subject area each teaches and to the awareness of the importance of education.

To the majority of us in education, the educational system and its ingredients all add up to the most effective educational program this world has ever known and I believe that.

We feel an urgency to know, to understand, to solve, to innovate, to accomplish. We know that society has identified education as the prime curator of our chaotic problems. We are caught in a revolution of rising expectations; and as we examine the problems, we see them as massive and diverse and increasing and demanding and we increase our efforts.

Mathematics teachers try to do two things at one time. They try to protect the identity and image of the subject matter. (1) You are disciplinarians of your subject matter. You must have a commitment. (2) You try to relate that to the students who wander into your classroom.

What, if anything, will provide the point of departure to resolve the differences between reality and expectations.

Education is a societal institution and as such its structure resists change. We were created to protect, contain, perpetuate the structure of our society and of our system. That particular society no longer exists. As society has changed, education has tried to meet changing needs.

Modern mathematics was math for the masses. What is mathematical fluency? We should make the definition a little more realistic. What is it?

Nothing is more unequal than equal education. All students do not need the same mathematics.

Different students need different mathematics. Many do not need more than the basic computational skills. I call that a core fluency of mathematics.

For others, they need a secondary fluency of mathematics. They are going to be thinking and planning and using mathematics as an organizing force for fields outside of mathematics.

For others, they need a primary fluency because they have a feeling for and a commitment to mathematics. They want their lives to revolve around the abstractions; they want mathematics to define for them truth and reality.

Finally we should use modern mathematics as it was intended to be used when it was young. Modern mathematics was easily defined as a way of thinking, a way of seeing.

Seeing is believing. Believing is seeing. If you believe in understanding and if you believe that knowing how things function will cause you to retain it, you can see the purpose for modern mathematics.

It is not too late to save a very grand and glorious innovation. It is our primary task in American society to teach children. We must teach children mathematics.

THE VIN'S: ZERO AND ONE

Sister Mary Petronia Van Straten, SSND
Mount Mary College
Milwaukee, Wisconsin

It is quite certain that almost everyone knows about the VIP's, but do they know what VIN's are? Make an educated guess! You are right. The VIN's are VERY IMPORTANT NUMBERS: namely, zero and one. It is very true that these numbers play a very important role in mathematics. It is amazing what a substantial bit of mathematics one would know, if one knew all the properties and characteristics of these two numbers and knew how to put them to use in particular situations.

Let us first consider the number zero. Many students have the misconception that "zero" and "nothing" may be used interchangeably, and they try to explain operations involving zero in those terms. This is a mistake! Though the dictionary defines nothing as the absence of all magnitude or quantity and also as zero, mathematically speaking there is a decided distinction. To illustrate the difference a student who has not registered for a course has absolutely nothing in that course, but before the student could hope to obtain a grade of zero, he would have to register for the course. At the elementary level, when one considers the meaning of a digit in a numeral, such as 205, one should say there are *no tens* or *not any tens*, rather than say or permit students to say there is nothing in ten's place.

Zero is a perfectly good number and indeed special! Actually it is a *whole number* because it is the cardinal number of the empty set. Zero is an *integer*. When one draws a picture of the number line, he arbitrarily chooses some reference point, zero and some convenient unit of length. Then he uses this unit of length to locate a point that is one unit to the right of zero. With this point he associates the integer, $+1$. Proceeding in this manner, one locates points corresponding to the integers: + 2, + 3, + 4, and so on. In a similar way, one can locate points to the left of zero and associate with them the integers: – 1, – 2, – 3, and so on. Then one can have all the points on the line corresponding to the integers. Note that the number zero itself is neither positive nor negative. One might say it is a neutral element. Zero is an *even integer*. There are a number of ways to show this, as was pointed out in an article entitled, "Zero Is An Even Integer", written by Betty Plunkett Lichtenberg and published in the November, 1972 issue of THE ARITHMETIC TEACHER. Zero is a *rational number* because it can be expressed as the ratio of two integers: that is, zero over any non-zero integer. It can be written as 0/5 or 0/13 and the like, thus showing that zero has an infinite number of names in its rational number dress. Zero is a *real number*, since it belongs to the set of all decimals. Zero is a *complex number*, because it can be expressed as a + bi, where a and b are real numbers and $i^2 = -1$. Named in this way, it would be 0 + 0i.

Let us next consider how zero behaves in the four basic operations. In addition it plays a very unique and important role. When zero is added to any given number or any given number is added to zero, the given number does not lose its identity. Symbolically, n + 0 = 0 + n = n for every real number n. No other real number behaves that way. For this reason we give zero a very special title: the *additive identity element*.

The question might arise as to whether zero plays the same role in subtraction. However, without much effort, it becomes apparent that it does not, since 7 – 0 = 7, but 0 – 7 = – 7. In general, for every real number n, n – 0 = n but 0 – n = – n. Yet there is something a bit special here. When any given number is used as the minuend and zero is the subtrahend, as in n – 0 = n, then the given number does not lose its identity. Therefore, mathematically speaking, we would say that zero is a right identity element but not a left identity element with respect to subtraction. Note, however, that zero is not an identity element, because it does not work on the right and on the left. Zero plays another role in subtraction. Whenever the minuend and the subtrahend name the same number, the difference is zero. In symbols: n – n = 0 for all n.

How does zero behave in multiplication? If we restrict ourselves to whole numbers, those numbers which tell us how many elements there are in a set, then it makes good sense to think of multiplication (except for 0 x n) in terms of repeated addition. For example, two threes means 3 is taken as an addend 2 times or in symbols: 2 x 3 = 3 + 3 = 6. 5 x 3 = 3 + 3 + 3 + 3 + 3 = 15. In a similar manner, five zeros would mean 5 x 0 = 0 + 0 + 0 + 0 + 0 = 0. But no matter how many times zero is taken as an addend, the sum will be zero. In this way, we can show that any non-zero whole number times zero is equal to zero. What about 0 x n? To take n zero times as an addend does not make good sense, but we can make use of the commutative principle which holds that n x 0 = 0 x n. Since n x 0 = 0, likewise 0 x n = 0. In general, any whole number times zero or zero times any whole number is equal to zero. This is especially helpful for children at the elementary level, for it helps them to learn nineteen basic facts in mutiplication all in one stroke. What a bonus! We have not proved that for every real number n, n x 0 = 0 x n = 0, but what we have shown is satisfactory for the elementary level. Later in mathematics, perhaps in abstract algebra, students can prove, that given a ring with the operations of addition and multiplication, whose additive identity is zero, that for every n belonging to the ring, n x 0 = 0 x n = 0. Actually when students reach that level, they wonder why they should have to prove so obvious a statement as n x 0 = 0 x n = 0. At the elementary level children often find difficulty in a multiplication involving zeros, as in 205 x 372. But if they understood the special property of zero shown above, they should not have any trouble. In finding another name for 205 x 372, it would be helpful for them to think of 205 as 200 + 5. The zero for them in ten's place simplifies the whole process. All the child need do is to find 5 x 372 and then 200 x 372 and add the two products. A form in which to record this might be:

372	372	372	1860	372	
205	5	200	74400	205	
	1860	74400	76260	1860	→ 5 x 372
				74400	→ 200 x 372
				76260	

A very useful and much-used theorem which holds in any algebraic system with no divisors of zero is this: The product of two numbers is zero if and only if one or both of the numbers is zero. In symbols: ab = 0 if and only if a = 0 or b = 0. To cite one of the commonest examples of the use of this theorem, suppose $x^2 = 4$. Then $x^2 - 4 = 0$ or (x + 2) (x – 2) = 0. In this latter form the two numbers named have a product of zero. Therefore (x + 2) must be zero or (x – 2) must be zero. Consequently x is equal to 2 or x is equal to a –2.

Lastly what about the role of zero in division? This always seems to be the trouble spot for children and also for prospective elementary teachers. Three cases must be considered: zero as the dividend, zero as both dividend and divisor, and zero as the divisor. It seems that the only reasonable way to explain division involving zero is in terms of the basic definition of division, since division is the inverse of multiplication. To divide a number A by a number B means to find a number C such that C x B = A. It is understood that, if the division is to be well-defined, two things are necessary: (1) that a number C exists — that is, one must be able to find such a number, and (2) that the number C is unique — that is, it must be the one and only number that fits the situation.

First, suppose the dividend is zero and the divisor is not zero. Take the particular example 0 divided by 5. What does that mean? It means we must find a number C such that C x 5 = 0. It is evident that such a number C, zero in this case, exists and is unique. It is the only number which will make the above open sentence true. Hence in this case division is well defined. Zero divided by any number which is not zero is zero.

Second, suppose the dividend and the divisor are both zero, then what? Zero divided by zero equals what? We must find a number C such that C x 0 = 0. That seems simple enough, as it is easy to find such a number C. In fact we could replace C by 2 or 3 or 5, and the open sentence would be a true statement. Pursuing this further, one can readily see that C could be replaced by any number. So in this case a number C does exist but it is not unique. Any number would do. For this reason, this is not really considered a good division, and in mathematics it is often referred to as the indeterminate case.

Thirdly, suppose the dividend is not zero and the divisor is zero. Take the particular example, 7 divided by O = ? That means we must find a number C such that C x 0 = 7. In this case, no such number C exists, as we learned before that any number times zero is always zero and never 7 nor any other number which is not zero. Consequently it is impossible to divide by zero. If this is not a convincing argument, you might try repeated subtraction. Start with 7 and keep subtracting 0 until you obtain 0.

Much of mathematics is simply renaming numbers and being smart enough to find that particular name for a number that best fits the situation or simplifies it. Let us cite a few examples in which zero is renamed or is used as the additive identity.

One case of renaming zero is in one of the ways of showing that a negative integer times a negative integer is a positive integer. This procedure assumes that students have already learned that a positive integer times a negative integer or a negative integer times a positive integer produces a negative product. For example, $(-3)\ (+5) = -15$.

Then one proceeds as follows:

(1) $(-3)\ (-5) = ?$

(2) $(-3)\ (0) = 0$

(3) Rename zero as $(-5) + (+5)$.

(4) Then $(-3) \times [(-5) + (+5)] = 0$

(5) Using the distributive principle:

$(-3) \times [(-5) + (+5)] =$

$(-3)\ (-5) + (-3)\ (+5) = 0.$

(6) Therefore (some number) $+ (-15) = 0$.

(7) The only number that will make that statement true is $(+15)$.

(8) Consequently $(-3)\ (-5)$ must equal $+15$.

This example shows the renaming of a number by the use of zero as the additive identity element in the proof of the theorem stated previously: Given a ring with the operations of addition and multiplication and whose additive identity is zero, then for every element r belonging to the ring, $r \times 0 = 0 \times r = 0$.

The proof is as follows:

(1) $r \times r = r \times (r + 0)$—Renaming r using the identity element

(2) $= r \times r + r \times 0$—Distributive principle

(3) Since $(r \times 0)$, when added to $(r \times r)$, does not make $(r \times r)$ lose its identity, and the additive identity is unique, $(0 \times r)$ must be another name for the identity element.

(4) Therefore $r \times 0 = 0$.

(5) $r \times r = (r + 0) \times r$—Renaming r, using the additive identity

(6) $r \times r = r \times r + 0 \times r$—Distributive principle

(7) Again, as in step (3), $0 \times r$ is acting as the identity element and must be equal to zero.

(8) Therefore $r \times 0 = 0 \times r = 0$.

In deriving some of the formulas for finding derivatives in the calculus, one makes use of other names for zero, names like $f(x)g(a) - f(x)g(a)$ or $f(a)g(a) - f(a)g(a)$. So one can readily see that even in higher mathematics zero plays a prominent role.

Let us now turn our attention to the number one, which like zero, is a perfectly good number and special! The number one is a *counting number* because it is the first number we use when we count the elements in a set. It is a *whole number* because it is the cardinal number of any set which contains one and only one member. It is an *integer* and an *odd integer* as it leaves a remainder of one on division by two. The number one is a *rational number*, because it can be named as one 2/2 or 5/5 or in an infinite number of other ways. The number one is a *real number*, since it belongs to the set of decimals. The number one is a *complex number* because it can be written in the form 1 + 0 i.

How does the number one behave in the operations of addition and subtraction? In addition, adding one to a whole number gives the next whole number. Subtracting one from a whole number give the preceding whole number, except in the case of zero. If one is subtracted from zero, a negative one is obtained.

In multiplication one plays a very unique and important role. When any given number is multiplied by one or one is multiplied by any given number, the given number does not lose its identity. Symbolically, n x 1 = 1 x n for every real number n. No other number behaves that way. For this reason we give the number one a very special title: the multiplicative identity element.

Does the number one play the same role in division? The number one plays a role in division analagous to the role zero plays in subtraction. This is true because division is the inverse of multiplication and subtraction is the inverse of addition. When any given number is used as the dividend and one as the divisor, as in n divided by 1, then the given number does not lose its identity. However, if the dividend is one and the divisor any given non-zero number, the quotient is not the given non-zero number. So, mathematically speaking, we would say that the number one is a right identity element for division but not a left identity. Therefore, it cannot be called an identity element. Whenever the dividend and divisor name the same non-zero number, the quotient is the number one. In symbols, n/n = 1 for every number n which is not zero.

One, under various names, is frequently used to simplify expressions or processes. One example of this is in the addition and subtraction of fractions. For example, in adding 2/3 and 1/4, we would rename 2/3 as 8/12 and 1/4 as 3/12. Actually, in this renaming, we are multiplying both 2/3 and 1/4 by the number one, using the name 4/4 for the number one.

In division of fractions, the principle that the dividend and the divisor may be multiplied by the same non-zero number without changing the quotient and the special role of the number one can be used to explain the inversion method, "invert the divisor and multiply". Take 3/4 ÷ 2/3 as an example. If we could in some honorable and honest way find a way to make the divisor one (and we can!), we know that the quotient would be the same as the dividend. If we multiply 2/3 by its reciprocal 3/2, then we obtain the number one. But if we multiply the divisor by 3/2, then we must also multiply the dividend by 3/2. So we have: 3/4 ÷ 2/3 = (3/4 x 3/2) ÷ (2/3 x 3/2) = (3/4 x 3/2) ÷ 1 which equals 3/4 x 3/2 = 9/8 or 1 1/8.

Another use of the number one in the area of division of fractions might be illustrated. To divide one fraction by another, say 8/9 ÷ 2/3, one could simply say, 8 ÷ 2 = 4 and 9 ÷ 3 = 3. Therefore 8/9 ÷ 2/3 = 4/3 = 1 1/3. Is this a correct procedure? Yes, it is, as it checks if one uses the basic definition of division. Does it work every time? It does! But the truth is there are times when it does not prove as simple as in the case above. In fact, in the majority of cases it does not prove to be that simple. Yet it can be done. For example, take 5/8 ÷ 3/4. At a glance one can see that one would like to have the factor 3 as part of the number named in the numerator of the fraction which is the dividend. By using 3/3 as a name for the number one, one can supply 3 as a factor: (5/8 x 3/3) ÷ 3/4 = $\frac{5\times3}{8\times3} \div \frac{3}{4} = ?$ (5 x 3) ÷ 3 = 5 and (8 x 3) ÷ 4 = 6. Therefore 5/8 ÷ 3/4 = 5/6. One can use this method of "doctoring the expression" in more complicated instances as in this example:

5/8 ÷ 2/3 = 5/8 x (2 x 3)/(2 x 3) ÷ 2/3

$$\frac{5\times2\times3}{8\times2\times3} \div \frac{2}{3} = \frac{15}{16}$$

There are many other examples in elementary and advanced mathematics where complex expressions can be reduced to simple ones by multiplying by the number one, using that special name for one that best fits the situation. To cite just one example and to provide another and perhaps the simplest method of dividing one fraction by another (it might be called the complex fraction method), consider 5/8 ÷ 2/3. This can be written in the form we often use for division $\frac{5/8}{2/3}$ Then both the dividend and the divisor may be multiplied by the number one, named as 24/24. $\frac{5/8}{2/3} \times \frac{24}{24} = \frac{15}{16}$. Surely junior high school students of mathematics and those beyond that level could divide one fraction by another mentally, if they became apt at this method.

There are instances in mathematics where the number one, though not explicitly written, is understood to be. For example, in writing x, the coefficient of x is understood to be one, and could be written as 1 · x. When the number one is used as an exponent, it is generally not written but understood, which means that x could also be written as x^1.

Zero and one play an interesting role together. Any number, except zero, raised to the zero power is equal to one. It seems very logical that this be the case as can be seen in our base-ten numeration system where we can represent any number as a sum of terms, each of which is some basic symbol times a power of the base. For example,

$2537 = 2$ thousands $+$ 5 hundreds $+$ 3 tens $+$ 7 ones

$$= 2(1000) + 5(100) + 3(10) + 7(1)$$
$$= 2(10)^3 + 5(10)^2 + 3(10)^1 + 7(10)^0$$

Since the powers of ten are decreasing as one proceeds from left to right, it would seem reasonable that 10^0 should be another name for the number one. Applying the laws for exponents, one can justify that this is the case. b^n, which is read "b to the n^{th} power", means b taken as a factor n times. Then $b^3 \cdot b^2 = b^5$, because b^3 means b taken as a factor 3 times and b^2 means b taken as a factor 2 times, so all together b is taken as a factor 5 times, which is written b^5. In general, $b^m \cdot b^n = b^{m+n}$, where m and n are positive integers. What does b^0 mean? To say that b^0 means b taken as a factor zero times has no meaning. So we must find another way to define it, a way that will be consistent with the properties for positive integral exponents. Let us examine a few situations in which the zero exponent might occur. Suppose we asked the question: b^3 times (b raised to what power) is equal to b^3, or in symbols $b^3 \cdot b^? = b^3$. If we wanted the same property of exponents to hold as above, $b^?$ would have to equal b^0, since $3 + 0 = 3$. But here b^0 is also behaving like the multiplicative identity element and since that element is unique, b^0 must be equal to the number one. This definition of b^0 would also fit the other property for exponents, which says that $b^m/b^n = b^{m-n}$, where $m > n$ and b is not zero and can be verified by use of the basic definition of a positive integral exponent. A logical question to ask would be: what if m is equal to n? Then $b^n/b^n = b^{n-n} = b^0$. By the basic definition of an exponent, $b^n/b^n = 1$. Then by the transitive property: $1 = b^n/b^n$ and $b^n/b^n = b^0$ imply that $1 = b^0$.

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FRACTIONS

This is a teaching sequence for a class of ten and eleven years old boys from a variety of backgrounds in an English Private School. All have some acquaintance with the subject from primary school. Some can parrot the rules. Others have little idea. I decide to take the subject as a dialogue, picking out the ideas that I need, politely ignoring the others. The method is a familiar one in British schools. There is no rigour—we save that for later, much later. We are trying to secure the understanding appropriate at this age, based on experience and developed by reason. I have kept the form of the dialogue, as accurately reported as I can. I speak first.

Well, O.K., what is a fraction?

Part of something.

One over something.

Can anyone show me a fraction in this rooms? (Consternation, then someone tears a sheet of paper into two bits and waves them at me)

What fraction is that?

A half!

How do you know?

Because he folded it onto itself.

No, I didn't!

Well, you should of.

Why?

Because it makes two equal bits. Then they're halves.

Right, well done! But a half is a very simple fraction, can anyone show me another one—and without tearing up paper?

That window pane, its a quarter of the window.

Are there any other fractions there?

Those three panes, they are three quarters.

Two quarters. (Pause) *But that's a half window* (Long pause)

This is the moment that I am waiting for. I believe that the whole of the subject can be made to rest on two ideas, and both have now been mentioned. The first, primitive one is derived from experience. Sheets of paper can be folded or cut into halves, quarters, thirds, and so on. Two halves, four quarters, three thirds, etc., recombine to make the whole. Experiments with weights and measures reinforce this. Then on to abstract ideas, seven days in a week, so sevenths or four fives in twenty, so quarters. The second basic idea is that of equivalence. This takes longer.

A simple diagram make it clear why it is true that $\frac{2}{4} = \frac{1}{2}$

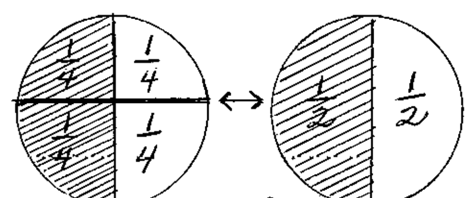


Fig.1

in a special sort of a way. Two quarters of a cake are not *the same* as one half. The cake has been divided in a different way, more cuts with a knife are required. Some crumbs have been lost. In an idealized sort of way they are equivalent, it is

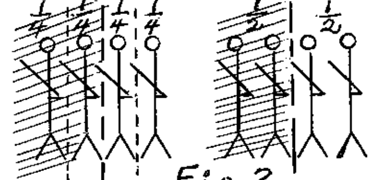
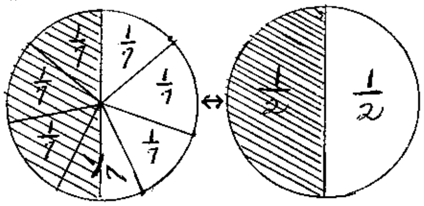


Fig.2

easier to see it with toy soldiers (Figure 2). But it remains a deep idea; and it is not obvious to a child that a fraction of a cake and a fraction of four soldiers can be represented by the same mathematical symbol.

Then it is necessary to make things more complicated



$\frac{3\frac{1}{2}}{7} = \frac{1}{2}$;

Figure 3

and seven soldiers will not do—try asking why?

I find it necessary to set the boys to work drawing, cutting and writing to make certain that this stage was not omitted by primary teachers eager for results. There are some muttered comments that this is baby-ish. But there is more to come.

Fractions can be *complicated*, as well as simplified. Some boys have a superstitious dread of two-quarters (two-fourths) or five-tenths and think them illegal. But I need them to be as happy saying one-half is equivalent to five-tenths as when saying that five-tenths is equivalent to one-half. So I define the meaning of the expression 'complicating a fraction' and we go on talking.

How many different complications of $\frac{2}{3}$ are there?

Lots, let's see $\frac{4}{6}, \frac{8}{12}, \frac{16}{24}$, *... thirty-two over ...*

You have missed out $\frac{6}{9}$.

Thirty-two over forty-eight and ... I can't do any more but I could go on and on, there's lots and lots, ... any number.

What is the rule for complicating them?

You just take twice times the top (numerator) *and bottom* (denominator).

How do you get $\frac{6}{9}$, then?

Well, you can take three times two.

And five times, like $\frac{1}{2} = \frac{5}{10}$

So the rule is, 'You can multiply the top and bottom by two or three'?

Yes.

No, and five.

So, take $\frac{3}{4}$, you say twice the top, 6, and three times the bottom, 12 and then $\frac{3}{4} = \frac{6}{12}$?

(Howls) *No! That's stupid!*

Well, what's the matter with it?

Top and bottom by the SAME thing.

Good, what's the rule then?

'You can multiply top and bottom by the same number, two or three'

Or five.

Does that make sense?

Not really; you had better say it ...

How about, 'You can multiply top and bottom by the same number, providing it is 2, 3 or 5'?

Or 6.

Or 7 or 8.

So some one writes on the blackboard, 'A fraction is equivalent to the fraction obtained by multiplying top and bottom by the same number.' (No one thought of 0) General agreement, mutual congratulations and the class settle down happily to finding out who can produce the most horrible complications of a simple fraction. One boy produces $\frac{1}{2} = \frac{1\ 379\ 468\ 213}{2\ 758\ 936\ 426}$.

Some arrive for checking without the original fractions; how would you like to be asked to check $\frac{370\ 368}{864\ 192}$? I then asked for both the two fractions and the top and bottom multiplier to be given and we produced the format used in the texts of the School Mathematics Project, which we are using. This looks like this: $\frac{1}{3} \xrightarrow[\times 4]{\times 4} = \frac{4}{12}$

We also complicated 2 into $\frac{8}{4}, \frac{64}{32}$, etc. but no one ventures into a fractional multiplier, even after the example pictured in Figure 3 above.

Let's come back to some easy ones and think of doing something slightly different with them. Do you agree that $\frac{8}{12} = \frac{2}{3}$?

Yes.

What have I done this time to top and bottom?

Divided by four.

Right. Could I have obtained the same answer by multiplying by something?

Huh?

What must you multiply 8 by to get 2?

A quarter?

...Excellent!.... $\frac{8}{12} \xrightarrow[\times \frac{1}{4}]{\times \frac{1}{4}} = \frac{2}{3}$

A quarter *of* 8 means a quarter *times* 8, doesn't it?

I don't quite see that, Sir!

The translation of 'of' into 'multiplied by' is not easy. The cure is probably to go back to structured materials, or pebbles.

```
000   000   000   000        4 groups
000   000   000   000   ->      of       ->  4 × 6 = 24.
                                six

While   oooooooo             a half
        oooooooo        ->      of       ->  ½ × 16 = 8.
                             sixteen
```

But for the moment we press on with a whole new range of nasty complications of harmless fractions, top and bottom being multiplied by $\frac{1}{3}$, then $\frac{3}{4}$, then $13\frac{2}{3}$, etc. Some of the class are not very good at these, the error level rises, but we hope that this will clear up later.

Some of the complications produced are 'four-deckers', like this:

$$\frac{2 \quad \xrightarrow{\times 1\frac{1}{4}}}{3 \quad \xrightarrow{\times 1\frac{1}{4}}} = \frac{2\frac{1}{2}}{3\frac{3}{4}}$$

There were no complaints, but, accepting that the boys know what fifths and sevenths are, what are they to mean by 'three-and-three quartersths'? Is it time to interpret fractions as division sums.

Is one third a number?

No, not really.

It's part of a number.

It's not like 1, 2, 3, but it must be a number.

Usually I give you a problem and you tell me the answer. Let's do it the other way round! The answer is 6. What was the problem?

2 × 3.

3 + 3.

And 2 + 4 and 1 + 5 and . . . (They soon get the idea!)

Lots of problems have 6 for an answer. Now the answer is one-third. What was the problem?

1 + 3.!!!

This is more sudden than I had expected and sounds like a parrot response. Does the class know, or does the boy know what he is talking about?

When I give you the problem 12 ÷ 3, you have to work something out in your head to get the answer, 4. If I give you the problem 1 ÷ 3, and you give me the answer, $\frac{1}{3}$, you have not had to work anything out, have you? How is that?

Huh?

Well, you have just written down the question in another form, haven't you? (On the blackboard) $1 \div 3 = \frac{1}{3}$. What is the answer to the problem 123 ÷ 345?

It's $\frac{123}{345}$, yes, I see . . .

But they could be worked out—by decimals.

Indeed they could, that's another story though.

A fraction, then, is both a problem and its answer. It can be the end of the problem, or the start. There is no way of avoiding this difficulty of the notation of fractions that I know, apart from familiarity and frequent use.

Can you mark out a length of 3 metres?

Yes, easy!

Weigh up 3 kilograms of potatoes?

Yes!

Measure out a length of $\frac{1}{3}$ metre? Weigh up kilogram?

Yes, of course.

Do you think that $\frac{1}{3}$ is as much a number as 3?

Yes.

It's much smaller, though.

It's point three three three three, . . .

Now the ideas and the experience required to master them are available. We have talked about the nature of fractions, seen how they arise, drawn pictures of them and have studied equivalence. The time has come to start working with them—and this will strengthen the conviction that fractions are numbers in their own right, that they do not have to be 'worked out' as decimals.

Can you add $\frac{2}{7}$ to $\frac{3}{7}$?

Easy!

What about adding $\frac{4}{7}$ to $\frac{5}{7}$?

That's $\frac{9}{7}$.

Can you have more than seven-sevenths then?

Of course, seven-sevenths is one, this is two more, $1\frac{2}{7}$.

Can you add $\frac{2}{7}$ to $\frac{3}{8}$?

(After a pause) *You want a common denominator.*

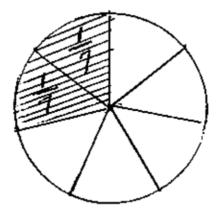
What's that and what for?

It's 2 × 3, no I mean 7 × 8. I don't know what for.

You say 2 × 8 and 3 × 7.

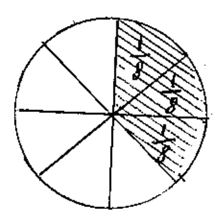
Why?

Well, don't let's guess, let us have a look at some diagrams. What fraction does this represent?



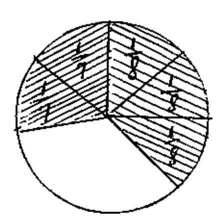
That's $\frac{2}{7}$

And what is this?



That's $\frac{3}{8}$

Well, let's put them together and see what we have got.



What have we got here?

(Doubtful noises) *We can't tell.*

If you can use this for $\frac{2}{7}+\frac{3}{7}$, why not for $\frac{2}{7}+\frac{3}{8}$?

They're different.

Excellent! So they are. Could we not make them the same?

Yes, yes, complicate them!

Into what?

56's. I told you, the common denominator!

I know you did, and I didn't say you were wrong, but you couldn't tell me what you wanted it for. Now make them both 56ths.

So we go to the blackboard and one boy writes

$$\frac{2}{7} \begin{matrix} \xrightarrow{\times 8} \\ \xrightarrow{\times 8} \end{matrix} = \frac{16}{56}$$

and another

$$\frac{3}{8} \begin{matrix} \xrightarrow{\times 7} \\ \xrightarrow{\times 7} \end{matrix} = \frac{21}{56}$$

And then we write $\frac{2}{7}+\frac{3}{8}=\frac{16}{56}+\frac{21}{56}=\frac{37}{56}$. It is a small step to see that the same thing enables you to subtract fractions with different denominators. I do not suggest the notation

$$\frac{2}{7}+\frac{3}{8}=\frac{16+21}{56}$$

One boy is working his examples for practice like this (The class is now setting itself additions and subtractions and solving them). I ask him what he thinks he is doing, and he replies 'saving ink'. A convincing answer and I do not interfere. After a bit we move on to $5\frac{2}{7}+4\frac{3}{8}$. We treat this as $5+4+\frac{2}{7}+\frac{3}{8}$. I do not mention associative or commutative laws. Save these until counter examples make it important to see whether or not they are true. For $5\frac{2}{7}-4\frac{3}{8}$, we convert first to $4\frac{9}{7}-4\frac{3}{8}$, and then go ahead. There is little difficulty.

Today we shall investigate the multiplication of fractions. Starting with something simple, what is $5\times\frac{2}{7}$? $\frac{10}{7}$

How did you get that?

I said five times two is ten

Five time two whats is ten whats?

Sevenths.

Right. Sevenths as numbers, or seventh parts of things, or indeed anythings combine in this same way, don't they? So what is the rule for multiplying a fraction by 5?

Multiply its top by 5.

Why not its bottom as well?

That would leave it the same, complicated.

Yes, of course, that is the rule for finding an equivalent, complicated, fraction. Why not just multiply the bottom by 5 then?

Because it's not the right way to do it.

It would make it into thirty-fifths, that's no good.

No good for multiplication, certainly, but no good for anything? Let's have a look. (On the blackboard)

$$\frac{2}{7} \xrightarrow{\times 5} \longrightarrow \frac{2}{35}$$

What has happened to $\frac{2}{7}$?

It's divided by 5!

Taken a fifth of it

Yes, I like the second way of thinking about it, let's write it out again in proper notation. How did you know that my previous try was not good mathematics?

Because you put an arrow, not an equals. (This referred to some previous work, when I had described ' ' as a 'sacred' sign, never to be used unless it is absolutely true and unless the mathematics is absolutely right. When I just mean that something seems to follow I use the 'careless' sign $\longrightarrow$)

Well, here it is. (On the blackboard)

$$\frac{2}{7}\times\frac{1}{5}=\frac{2}{35}$$

Now we'll try another one from a different angle. Work out $\frac{1}{3}\times\frac{5}{7}$. How will you do that?

Will you turn it upside down?

What on earth for? What did we say that multiplying by $\frac{1}{3}$ was the same as?

Dividing by 3.

Can you divide $\frac{5}{7}$ by 3?

Not very well.

Can you complicate it into something that will divide by 3?

Yes, times it by 3 (Horrible expression!)

$$\frac{5}{7} \begin{matrix} \xrightarrow{\times 3} \\ \xrightarrow{\times 3} \end{matrix} = \frac{15}{21}$$

$$\text{So, } \frac{5}{7}\div 3=\frac{15}{21}\div 3=\frac{5}{21}.$$

Very good, but I want it in the multiplication form with which we started; and you seem to have got things back to front. Are you sure that $\frac{1}{3}\times\frac{5}{7}$ is the same as $\frac{5}{7}\times\frac{1}{3}$?

They were sure, but not clear why they were sure. I left this to be cleared up later and handed one of the boys the chalk to rewrite the previous boy's division sum. He wrote

$$\frac{1}{3} \times \frac{5}{7} = \frac{1}{3} \times \frac{15}{21} = \frac{5}{21}$$

Now calculate $\frac{3}{4} \times \frac{5}{8}$. Treat this as $3 \times \frac{1}{4} \times \frac{5}{8}$. Argue it out for us to hear.

Well, a quarter of five eighths you can't. So complicate it by four. Five-eighths is twenty thirty-twoths (thirty-seconds), *and a quarter of this is five thirty-twoths.*

Let me write this up, with the other bit:

$$\frac{3}{4} \times \frac{5}{8} = 3 \times \frac{1}{4} \times \frac{5}{8} = 3 \times \frac{1}{4} \times \frac{20}{32} = 3 \times \frac{5}{32} = \frac{15}{32}$$

Now look at all these results and find me the quick way of doing it:

$$\frac{2}{7} \times \frac{1}{5} = \frac{2}{35}; \frac{1}{3} \times \frac{5}{7} = \frac{5}{21}; \frac{3}{4} \times \frac{5}{8} = \frac{15}{32}$$

Yes; multiply the tops and multiply the bottoms; I remember now!

Right. That's the rule, but don't forget that we know how to do it from first principles. Now set yourselves some multiplications, include mixed numbers like $2\frac{3}{4}$, and try particularly some like $\frac{2}{3} \times 1\frac{1}{2}$ and some like $\frac{3}{4}$ x 6.

It amuses me to see the class enthralled by simple repetitive work and determined to find some fraction problem that will beat their neighbours. I am saving fractions with zeros in them until later. But it is time to tackle division.

Remind us of the answer to 4 ÷ 6.

$\frac{4}{6}$, *I mean* $\frac{2}{3}$

Good. Now tell me another way of writing

$$\frac{2}{7} \div \frac{3}{8}$$

Well, I suppose you could put $\dfrac{\frac{2}{7}}{\frac{3}{8}}$

A short digression is needed here on the need for a long line for the middle fraction bar. Later we found three values for $\dfrac{\frac{2}{7}}{\frac{3}{8}}$ according to which bar was taken 'to be the long one'.

Now that's a pretty uncomfortable four-decker fraction. Could we complicate it to simplify it?

Multiply it by something?

Common denominator.

Multiply top by 7 and bottom by 8.

(Chorus) *No you can't!*

If we are going to multiply top and bottom by something it had better be the same something, hadn't it?

What about 56?

What about it indeed. Shall we try it?

So, on the blackboard we write

$$\frac{\frac{2}{7} \xrightarrow{\times 56}}{\frac{3}{8} \xrightarrow{\times 56}} = \frac{\frac{112}{7}}{\frac{168}{8}} = \frac{16}{21}$$

And then, before an unnecessary method sinks in too deep, I show them another way:

$$\frac{\frac{2}{7} \xrightarrow{\times \frac{8}{3}}}{\frac{3}{8} \xrightarrow{\times \frac{8}{3}}} = \frac{\frac{2}{7} \times \frac{8}{3}}{\frac{3}{8} \times \frac{8}{3}} = \frac{\frac{16}{21}}{1} = \frac{16}{21}$$

I also point out that, when they get good at it, they can leave the bottom line out altogether—to the great relief of the boy who had earlier exhorted me to turn something upside down'.

Needless to say this approach can be made more rigorous. As detailed it depends on the group having previous knowledge, of some sort. Many questions are begged—but at junior secondary stage this is hardly surprising! I hope that these candid camera shots of lessons will be of interest to Texas teachers. Even if the subject matter is neither new nor particularly exciting, the method and the (mostly) genuine dialogue may amuse.

Alan Tammadge
Sevenoaks School
Sevenoaks, Kent, England

"Now that we've worked the first problem in class—take the next nineteen for tomorrow's assignment."

Executive Appointments

Dr. Harry Bohan, Sam Houston State University, Huntsville, has been appointed to serve as vice-president of TCTM, representing elementary education. He replaces Dr. Ida Mae Heard, retired.

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