Vol. XX No. 2

TEXAS MATHEMATICS TEACHER

March 1973

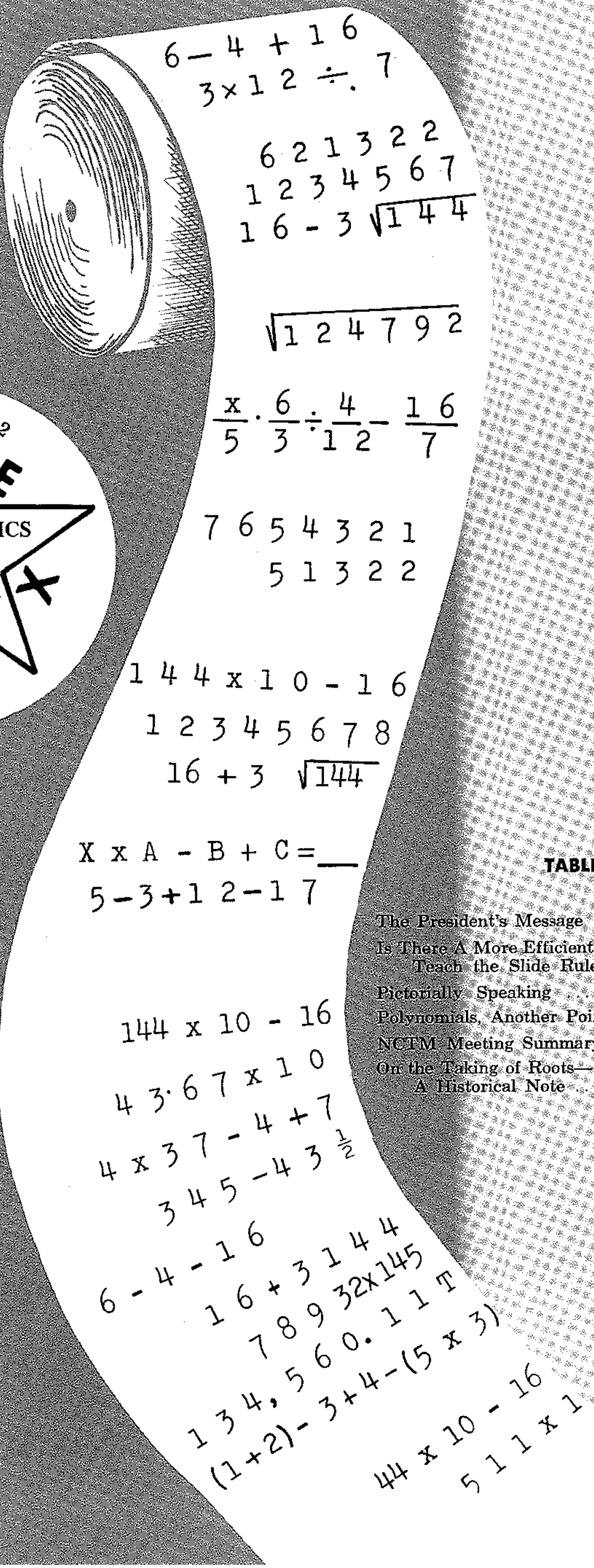
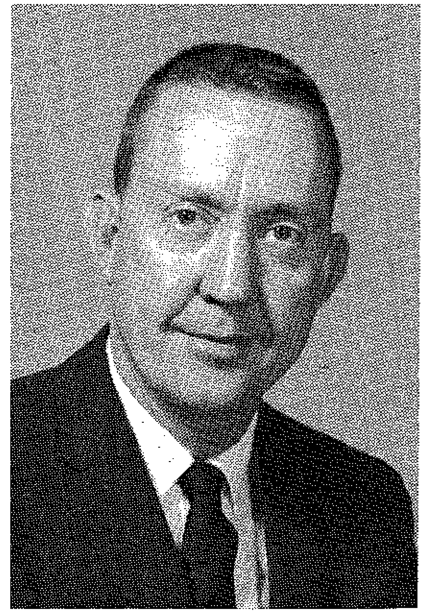


TABLE OF CONTENTS

TEXAS MATHEMATICS TEACHER is the official journal of the Texas Council of Teachers of Mathematics. The views expressed are the contributor's own and are not necessarily those of the publisher or the editor. All manuscripts and correspondence about this publication should be addressed to Mr. J. William Brown, *Texas Mathematics Teacher*, 100 So. Glasgow Drive, Dallas, Texas 75214.

TEXAS MATHEMATICS TEACHER is published quarterly by the Texas Council of Teachers of Mathematics. Payment of membership fee of $2.00 entitles members to all regular Council Publications.



PRESIDENT'S MESSAGE

by
James E. Carson
Immediate Past President

I have just returned from a very fine workshop at Gregory-Portland High School. It was co-sponsored by The Coastal Bend Council of Teachers of Mathematics and The Texas Council of Teachers of Mathematics. I wish to express my appreciation to Mr. W. C. Andrews, Superintendent of Schools, and Mr. T. M. McDonald, Principal, for making the buildings available to us and for their hospitality. I would also like to thank all of the textbook companies for making the refreshments available. I would like to thank Mrs. Shirley Ray and Mrs. Madge Simon and their co-workers for all they did in order to make it such an outstanding workshop. To the people who did such a fine job in their section meeting, I wish to commend you. The teachers really did like the workshop and it seems that this is the type of program that they prefer. Why not have some more throughout the State in the future? The officers of T.C.T.M. are willing to help in any way possible.

We will have had our annual meeting at C.A.S.M.T. by the time this journal reaches you. Mrs. Shirley Ray will be your new president and we are looking forward to two more great years with her very capable leadership.

May I take this opportunity to thank each and every one of you for the willing help and support that you have given me during my tenure of office. Without your help it would have been a tremendous task. I think there is something that T.C.T.M. can do for our mathematics teachers throughout the State and we need to explore further the things that our teachers want. I think by cooperating with local councils we can build a better mathematics program in every section of the State. I also would like especially to thank each officer of T.C.T.M. for his fine assistance.

The next thing we must think about is the annual convention of the National Council of Teachers of Mathematics to be held in Houston, April 25-28, 1973. We will be looking forward to seeing each of you there. Let's make it the biggest ever. We believe that we have something to offer in the Houston area and we invite each of you to attend some of our schools. They will be open for you to visit. COME TO HOUSTON IN APRIL.

In closing may I again say thanks for everything that each of you has done to make my tenure of president of T.C.T.M. an enjoyable one.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS MEETING

Houston, Texas

April 25-28, 1973

Make Your Plans To Attend

51st Annual Meeting, NCTM, April 25-28, 1973, Houston

Name-of-Site NCTM Area Meeting, August 15-17, 1973, Fort Worth

IS THERE A MORE EFFICIENT WAY TO TEACH SLIDE RULE?

by CHARLES TANNERY
Teacher, Pasadena Independent School District
Pasadena, Texas

There are two ways to teach mathematics. One is to take real pains towards creating understanding – visual aids, that sort of thing. The other is the old British system of teaching until you're blue in the face.

James Ray Newman

At times all teachers feel that they have been teaching until they are blue in the face, and a better technique would certainly be welcome. The following system for introducing slide rule has been very effective in participating schools for students from the seventh to twelth grade.

For a student who is learning the use of the slide rule, the most important goal should be accuracy. The way to improve accuracy is to use an instructional program with few and simple rules. If a single procedure is use time after time, the student masters the technique and forms a desirable habit. Here is an instructional procedure with this goal in mind.

Change all multiplication and division problems so that they will fit the pattern of n + 1 elements in the numerator and n elements in the denominator.

Example: $24 \times 51.3 \times 8450$

Rewrite: $\frac{24 \times 51.3 \times 8450}{1 \times 1}$

Example: $\frac{.0025}{81.6}$

Rewrite: $\frac{.0025 \times 1 \times 1}{81.6}$

Certainly multiplying or dividing by additional "ones" will not change the answer, and now any problem that contains multiplication and /or division can be performed using a single rule. Before explaining the rule, some terms will be renamed. There are three parts to a slide rule: the body, the slide, and the indicator or cursor. To make instruction more descriptive, one should refer to the slide as the "divider" and the cursor as the "mutiplier".

Using these terms, the rule for multiplying and/or dividing is to make the first setting with the multiplier, the next setting with the divider, and alternate their use throughout the problem. The final step will always be to use the multiplier since the problem has been constructed to have n + 1 elements in the numerator. The answer will always be found under the hairline after the last setting. For all problems, then, the first setting and the answer will be on the D scale, and *all* other setting will be made on the C scale.

Now one could consider the advantages of introducing the slide rule to students in this manner. Every time they pick up the slide rule, they will always move the multiplier to a number in the numerator first; then they will *always* move the divider to a number in the denominator next. With a relatively short period of practice, the procedure will become automatic, and this automatic reaction will eliminate most careless errors.

In determining the decimal, many good methods are available. One method involves using each number's characteristic and observing the use of the right index. The characteristic of a number refers to the exponent of the number when written in scientific notation.

Example: $362{,}000 = 3.62 \times 10^5$

The characteristic of 362,000 is 5

The decimal placement for the answer is determined by totaling the characteristic of each number in the problem. The student adds the characteristic of numbers in the numerator and subtracts the characteristic of numbers in the denominator. The other factor in determining the decimal placement for the answer is to note when the right index or more descriptively the "right one" was used. Each time the multiplier is set on the "right one," the total should be decreased by one. Each time the divider is set on the "right one," the total should be increased by one.

Example: $\frac{.0495}{.337 \times .889}$ The characteristic of:
.495 is − 2
.337 is − 1
.889 is − 1

total characteristic = $(-2) - (-1) - (-1) = 0$
Next the the effect of the "right ones" must be determined. When the problem is rewritten

$$\frac{.0495 \times 1 \times 1}{.337 \times .889},$$

it can be seen that multiplication by one will occur at two different times. Now by using the slide rule to actually multiply out the problem, one will be forced to use the left index for the first one and the right index for the second one. Using the left index has no effect but multiplying by the right index will deduct one from the characteristic total. Since the total was 0, the answer's characteristic will be $0 - 1 = -1$ and could be written as 1.65×10^{1}

If, in addition to accuracy, the teacher is interested in developing speed, he should consider how speed is increased when teaching other manual skills. In typing, for example, rhythm is a dominate factor, and rhythm indicates the key to building speed with the slide rule. When pressed to work more problems in a given time interval, it is a natural instinct to make the hand move more quickly resulting in a jerky motion. It is more effective to try to eliminate pauses between settings and to develop a smooth even pace. In order to obtain a constant pace one might try using a timer that makes a tone at regular intervals, for instance, every five to ten seconds, and having his students pace their settings by the tone.

Correct posture is also helpful in increasing accuracy and speed. One should encourage his students to hold their slide rules in a position where the problem can be read and the setting made with a minimum of head and eye motion. This will cut down on students' losing their place while working. Additional speed can be gained by keeping a running total of the characteristics instead of calculating the total after the problem is worked.

Hopefully some of these suggestions will be useful, but the real secret to efficiency on the slide rule is hard work. No matter how effective the teaching method becomes, no one can master the slide rule without practice.

(Reprinted due to errors on previous printing)

PICTORIALLY SPEAKING

by

DR. ADRIEN L. HESS

For a long time there have been people who insist that mathematics should mean only pure mathematics. To these persons mathematics is abstract and one must learn to think abstractly as early as possible. It is true that certain parts of mathematics can be understood only if the subject is considered abstractly.

Today many persons need to understand and use mathematics who do not need to consider it at the abtstract level. For many of these persons any aid to the learning and understanding of the subject would be welcomed. The printed page is available but this is not enough aid for many persons. If the idea or concept were also presented by means of a picture, drawing, or graph, understanding would come quicker and much time would be saved.

In the modern approach to teaching mathematics rote learning is to be replaced by understanding. Guessing and intuition are used as a means to discover solutions to problems. There are several avenues and approaches to learning things. Most people think in visual terms. Again, there are several approaches to learning through visualization. More than one of these approaches should be used, if necessary, to present an idea or concept. Some learn by one of the other senses, like feel, sound, taste, etc. These should not be overlooked.

There are many reasons for the utilization of pictures, graphs, drawings, etc., in considering a mathematical problem. Polya, in *How To Solve It*, emphasizes the drawing of a figure to help understand the problem. A picture or drawing is often an aid in discovering a solution to a problem. After finding a solution a picture or a drawing may point out certain restrictions or a second way to solve the problem may be seen. Pictures or drawings may lead to a rule or a generalization which can then be proved. They can help to interrelate algebra and geometry.

A picture or drawing can be used to make a result, a rule, or a generaliaztion seem intuitively evident. In arithmetic pupils are given the rule for the multiplication of two fractions. According to the rule the numerators are multiplied together and the denomnators are multiplied together. For example: $\frac{1}{4} \times \frac{3}{8} = \frac{1(3)}{4(8)} = \frac{3}{32}$. Consider the proper fraction $\frac{a}{b}$ as a unit which is divided into b congruent parts and a of these are taken. Consider the square in Figure 1 with a side of 1.

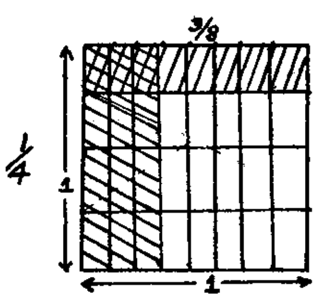


Figure 1.

Divide the square into 4 congruent rectangles. Shade in one of these. Again divide the square into 8 congruent rectangles, using a second side. Shade in 3 of these. The numerator of the answer is the number (3) of squares shaded both ways and the denominator is the number (32) of little squares in the unit square. For improper fractions it is helpful to make the side of the square have enough units so each fraction can be shown. Use a unit square for the namer or denominator of the fraction.

A drawing or picture can help prevent errors. A common error in squaring the quantity $a + b$ is to write $(a + b)^2$ as $a^2 + b^2$. Let each side of the square be $a + b$.

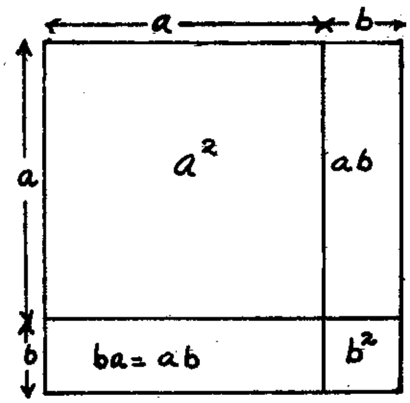


Figure 2.

The area of the square is $(a + b)^2$. The drawing shows two squares of area a^2 and b^2 respectively and two triangles of area ab each. Thus, $(a + b)^2 = a^2 + 2ab + b^2$.

A pictorial representation of an operation or procedure can often give meaning to what may seem to be a rote procedure. As an example of how a diagram may give meaning to an expression consider the statement "completing the square". This is an algebraic procedure which has more meaning when a geometric picture is given. Consider the rectangle of area $x^2 + 8x$ which is given in Figure 3. Consider the rectangle BCFE of area 8x.

Figure 3. (Page 7)

Divide this rectangle into two congruent rectangles. The rectangle GCFH encloses area of 4x. Place this rectangle so that C falls on D and F falls on E with GH below DE. Now, except for a missing part of $4 \times 4 = 16$, the result is a square $(x + 4)(x + 4)$. When this missing part is added the operation constitutes "completing the square". The steps used in making the geometric figure are exactly the algebraic steps in completing the square.

Another example of using geometric figure to give meaning to an algebraic algorithm is in taking the square root of a number N. One algebraic alorithm uses the identity $(a+b)^2 = a^2 + 2ab + b^2 = a^2 + b\,(2a + b)$.
This algorithm will be used to find $\sqrt{652}$.

		2 5.5
		$\sqrt{652.00}$
1)		400
2)	40	252
	45	225
3)	500	2700
	505	2525
		175

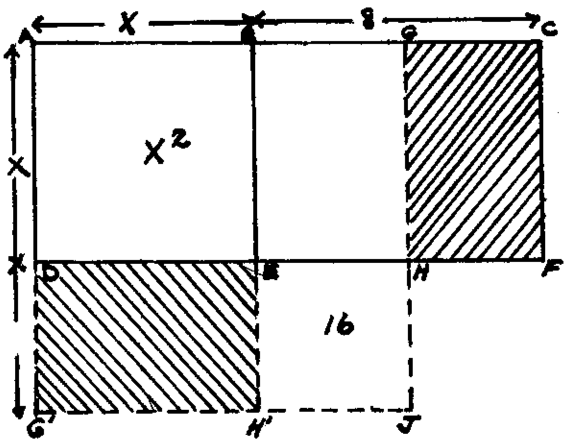


Figure 3.

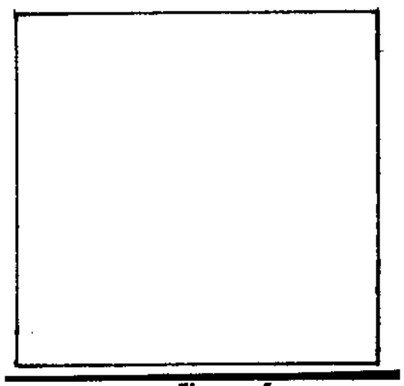


Figure 4.

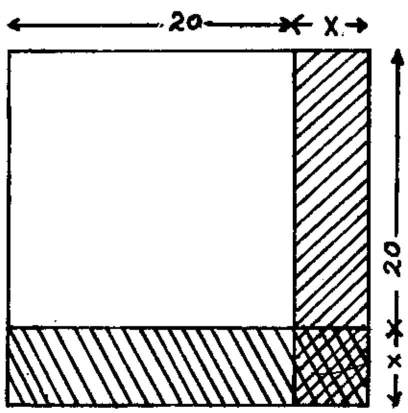


Figure 5.

Figure 4 is a square with area of 652. Part of this square consists of a square 20 by 20. This is shown in Figure 5.

The area of $20^2 = 400$ is the part that was removed in step 1 in the algebraic algorithm. The part not accounted for is the shaded parts in Figure 5. Let x represent the width of the area left. This area of 252 can then be represented by the rectangle in Figure 6.

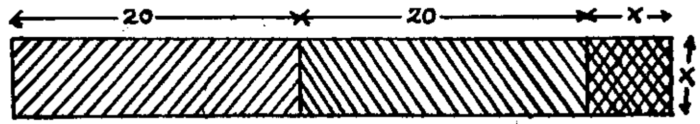


Figure 6.

From Figure 6 it can be seen that
$(20+x)^2 = 652$, $400 + 40x + x^2 = 652$, and $x(40+x) = 252$. This remainder is the first line in step 2 of the algorithm. A value for x must be found. An estimate can be obtained by dividing 252 (the area of the rectangle) by 40. 40 is an approximation to the length of the rectangle. In step 2 of the algorithm the first digit in the answer was doubled and then a zero digit was added. This was used as a trial divisor. When 252 is divided by 40, 6 looks like a reasonable answer. But $40 + x$ indicates that 6 must be added to 40 and when 46 is multiplied by 6 the product exceeds 252. Thus 5 is chosen and it is added to 40 and this sum of 45 is multiplied by 5. This is the same as was done in the second part of step 2 in the algorithm. When the product of $5 \cdot 45$ is subtracted from 252 the remainder is 27. This completes step 2 in the algorithm.

The area of 652 taken at the start can be represented as is done in Figure 7. From this figure it can be seen that

$$(25+y)^2 = 652,\ 625 + 50y + y^2 = 652,$$
$$\text{and } y\,(50+y) = 27.$$

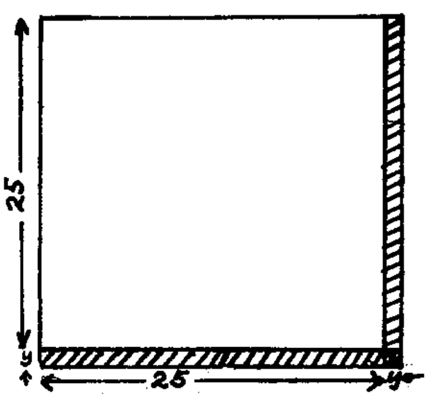


Figure 7.

Figure 8.

The rectangle in Figure 8 has an area of 27. If 27 is divided by 50 an estimate of 0.5 is obtained for y. For y(50+y) this gives 0.5(50.5) = 25.25. When this is subtracted from 27 a remainder of 1.75 is obtained. This coincides exactly with step 3 in the algorithm. To secure an answer correct to the hundredths the above steps can be repeated.

Another use of a digram or pictorial representation is to show a number of relationships together. An example of this is Figure 9 in which tangent ratios are shown. In a right triangle with one acute angle represented by α, tan α is the length of the side opposite angle α divided by the length of the side adjacent to angle α. Let OA be perpendicular to BC at C. Let $\angle$COB be any angle whose measure is not 45° and such that OC is of length 1. Let OK be perpendicular to OB, BE perpendicular to OB, EF be perpendicular to BE, LK perpendicular to OK. Now the non-right angles of the several triangles are either α or $(90 - \alpha)$.

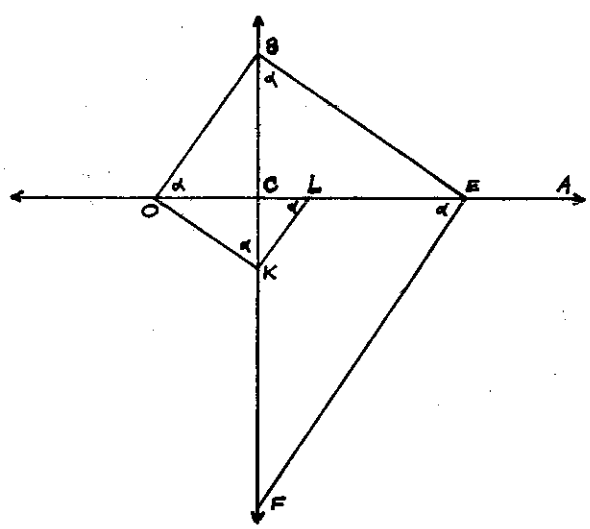


Figure 9.

In $\triangle OCB$ $\tan \alpha = \frac{\overline{BC}}{\overline{OC}} = \frac{\overline{BC}}{1} = BC.$

Therefore $\overline{OB}^2 = 1 + \tan^2\alpha = \sec^2\alpha.$

In $\triangle ECG$ $\tan \alpha = \frac{\overline{CE}}{\overline{BC}} = \frac{\overline{CE}}{\tan\alpha}$. Therefore $\overline{CE} = \tan^2\alpha.$

In $\triangle OKC$ $\tan \alpha = \frac{\overline{OC}}{\overline{CK}} = \frac{1}{\overline{CK}}$. Therefore $\overline{CK} = \cot \alpha.$

In $\triangle LCK$ $\tan \alpha = \frac{\overline{CK}}{\overline{CL}}.$

Therefore $CL = \frac{CK}{\tan\alpha} = \frac{\cot\alpha}{\tan\alpha} = \cot^2\alpha.$

In $\triangle FCE$ $\tan \alpha = \frac{\overline{CF}}{\overline{CE}}$

Therefore $\overline{CF} = (\tan\alpha)\overline{CE} = (\tan\alpha)(\tan^2\alpha) = \tan^3\alpha.$

In $\triangle FCE$ $\overline{EF}^2 = \overline{CE}^2 + \overline{CF}^2 = (\tan^2\alpha)$
$+ (\tan^3\alpha)^2 = \tan^4\alpha$
$+ \tan^6\alpha = \tan^4\alpha$
$(1+\tan^2\alpha)$
$= \tan^4\alpha \sec^2\alpha.$

There are many applications of equations of the type $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}$. Some of these are: the resistance of two conductors connected in parallel; the capacitance of two condensers connected in series; the relationship of the image, the object, and the focal length of lenses; the time it takes two men working on a task to do the work when it is known how long it takes each to do the work alone; the time it takes two pipes to fill a tank when both are running water into the tank or when one pipe is running water into the tank and the other is running water out. A graphical solution can be made of each of these applications.

Assume that one pipe alone can fill a tank in a hours and a second pipe alone can fill the same tank in b hours. How long will it take to fill a tanke if both pipes are running water into the tank?

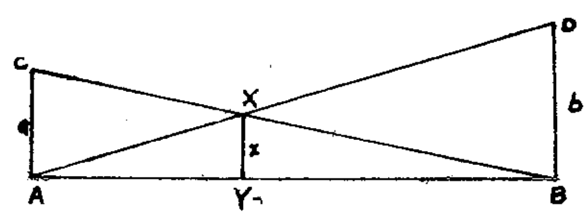


Figure 10.

Let AB be a line segment of arbitrary length. The length does not affect the solution. Let the line segments above AB be positive and line segments below AB be negative. Draw line segment AS of length a perpendicular to AB and line segment BD of length b perpendicular to AB. Draw line segments AD and BC to intersect at X. Let XY be perpendicular to AB. Then XY, of measure x, is the desired time. The proof is quite straightforward.

$\triangle AYX - \triangle ABD$. Therefore $\frac{x}{b} = \frac{AY}{AB}$

$\triangle BYX - \triangle BAC$. Therefore $\frac{x}{a} = \frac{BY}{AB}$

Hence, $\frac{x}{a} + \frac{x}{b} = \frac{AY}{AB} + \frac{BY}{AB} = \frac{AY + YB}{AB} = \frac{AB}{AB} = 1.$

Therefore $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}$.

The same method can be applied to the problem when one pipe is emptying the tank while the other is filling it. Again consider an arbitrary line segment AB with distances above this segment as positive and distances below it as negative. Figure 11 gives the solution.

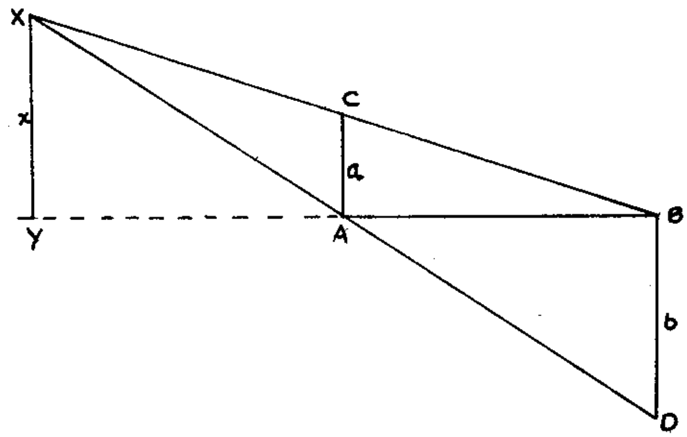


Figure 11.

It is quite easy to graphically multiply two positive numbers. Figure 12 gives a diagram to find a•b.

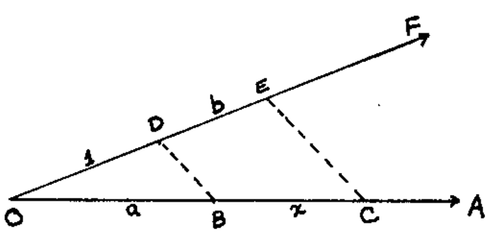


Figure 12.

Let OD a unit. Let the lengths of OB and DE be a and b respectively. Draw DB. Through E draw line segment EC parallel to DB. From porportional segments $\frac{1}{a} = \frac{b}{x}$. Therefore x = ab.

The same kind of diagram can be used to show a÷b by interchanging the unit and the length b.

A similar procedure can be used to show the product of two numbers with negative signs. In Figure 13 consider OA and OB as in a positive direction and OC and OD as in a negativve direction.

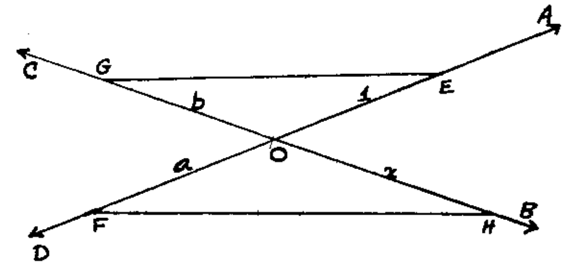


Figure 13.

Choose OE to be a unit. Let OG be of length b and OF of length a. For (—a)(—b) this is using the negative direction. Draw GE and through F draw a line parallel to GE to intersect OB in H. OH = x is the desired solution, and it is in the positive direction. Since △ GAE ~△ HOF, the corresponding sides are proportional. This gives $\frac{-b}{1} = \frac{x}{-a}$ and therefore x = (—a)(—b) = ab.

The diagrams given here are some that can be used to help visualize mathematical ideas, concepts and problems. There are other ways to visualize many of these. All of the examples have been in two dimensions, but they may be easily extended to three dimensions. Figure 13 can be modified to apply when one number has a positive sign and the other number has a negative sign.

Pictures and diagrams can be very helpful in finding a solution to a problem but they should not be regarded as a proof. A brief bibliography is included for anyone who wishes to do further reading.

BIBLIOGRAPHY

(1). Donovan Johnson and Gerald Rising—*Guidelines For Teaching Mathematics,* 1967, Wadsworth Publishing Co., Belmont, California.

(2). L. Clark Lay, *Arithmetic, An Introduction to Mathematics,* 1961, Macmillan Co.

(3). G. Polya—*How to Solve It,* 1945, Princeton University Press, Princeton, New Jersey.

(4). W. W. Sawyer—*Visions in Elementary Mathematics,* 1964, Penquin Books Inc., 3300 Clipper Mill Road, Baltimore, Maryland.

(Dr. Adrien L. Hess,
Montana State University)

POLYNOMINALS, ANOTHER POINT OF VIEW

by Louie C. Huffman
Midwestern University
Wichita Falls, Texas

In higher algebra polynomials are defined over rings, but the polynomials that are of interest in high school are usually restricted to the real numbers. I will therefore confine my remarks to polynomials over the reals.

Definition: Let $a_0, a_1, \ldots, a_n$ be real numbers and n be a non-negative integer. The function,

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

with domain the set of real numbers, is called a polynomial over the reals.

Examples: $2x^2 + 3x - 4,$
$-1/2x + 1,$
$\pi x^3 - 2x^2 + 4.1x + 7.$

Now suppose we had a "Polynomial Machine". This is something like a desk calculator except it "works on" polynomials instead of real numbers. The instrument panel (or place where you could see the polynomial you had fed into the machine) might look like this:

$$\bigcirc x^0 + \bigcirc x^1 + \bigcirc x^2 + \bigcirc x^3 + \bigcirc x^4$$

where the x^0, x^1, x^2, x^3, and x^4 are painted on the surface of the machine and numbers appear in the holes to denote different polynomials. Let's assume that two polynomials can be fed into the machine and that it can then be instructed to add or multiply the two polynomials:

	(2)	x^0	+	(1)	x	+	(-2)	x^2	+	(0)	x^3	+	(0)	x^4
	(3)	x^0	+	(2)	x	+	(4)	x^2	+	(0)	x^3	+	(0)	x^4
Add	(5)	x^0	+	(3)	x	+	(2)	x^2	+	(0)	x^3	+	(0)	x^4
Mult	(6)	x^0	+	(7)	x	+	(4)	x^2	+	(0)	x^3	+	(-8)	x^4

Notice that all we really fed into the machine were ordered n-tuples (4-tuples in this particular case). The x^0, x^1, etc. are painted on the surface of the machine. They had nothing to do with the addition and multiplication that took place. We found that

$$(2, 1, -2, 0, 0) + (3, 2, 4, 0, 0) = (5, 3, 2, 0, 0) \quad \text{and}$$

$$(2, 1, -2, 0, 0) \times (3, 2, 4, 0, 0) = (6, 7, 4, 0, -8).$$

Now this particular machine cannot be fed nor can it feed out polynomials of degree higher than four. But we can build a "mathematical machine", an "abstract machine", a mathematical system, that has no such limitation.

Let us take a different defintion for a polynomial:

Definition: If a_i, for $i = 0, 1, 2, \ldots$ are real numbers and for some non-negative integer k, $a_m = 0$ if $m \geq k$, then

$$(a_0, a_1, a_2, \ldots)$$

is a polynomial.

Examples: $(-1, 0, 3, 4, 0, 0, 0, \ldots)$ is a polynomial;

$(0, 0, 0, \ldots)$ is a polynomial;

$(1, 2, 3, 4, \ldots, n, \ldots)$ is <u>not</u> a polynomial;

$(0, 0, 0, 1, 1, 1, \ldots)$ is <u>not</u> a polynomial.

We see immediately that addition in this system should be defined as follows:

Definition: If p and q are polynomials and

$$p = (a_0, a_1, a_2, \ldots, a_n, \ldots) \quad \text{and}$$

$$q = (b_0, b_1, b_2, \ldots, b_n, \ldots)$$

then $p + q = (c_0, c_1, c_2, \ldots, c_n, \ldots)$

where $c_0 = a_0 + b_0, \; c_1 = a_1 + b_1, \ldots, \; c_n = a_n + b_n, \ldots$.

In other words to add two polynomials we add pointwise; add the first terms to get the first term of the sum, the second terms to get the second term of the sum, etc.

Addition in this system carried over easily from our experience with adding polynomials in the usual form. But multiplication is not so obvious. Let's look at another method of multiplying polynomials in the old form. Consider the example from above:

$$(-2x^2 + x + 2) \text{ X } (4x^2 + 2x + 3)$$

$$= (2 + x - 2x^2 + 0x^3 + 0x^4) \text{ X } (3 + 2x + 4x^2 + 0x^3 + 0x^4)$$

$$= 2 \cdot 3 + (2 \cdot 2 + 1 \cdot 3)\ x + (2 \cdot 4 + 2 \cdot 1 - 2 \cdot 3)\ x^2 +$$

$$(2 \cdot 0 + 1 \cdot 4 - 2 \cdot 2 + 0 \cdot 3)\ x^3 +$$

$$(2 \cdot 0 + 1 \cdot 0 - 2 \cdot 4 + 0 \cdot 2 + 0 \cdot 3)\ x^4$$

$$= 6 + 7x + 4x^2 + 0x^3 - 8x^4.$$

Using this method one can easily see how to define multiplication in this new system:

Definition: If p and q are polynomials and

$$p = (a_0, a_1, a_2, \ldots)$$

and $$q = (b_0, b_1, b_2, \ldots)$$

then $$p \text{ X } q = (c_0, c_1, c_2, \ldots)$$

where $$c_0 = a_0b_0, \; c_1 = a_0b_1 + a_1b_0, \; c_2 = a_0b_2 + a_1b_1 + a_2b_0,$$

$$c_3 = a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0, \ldots$$

$$c_k = a_0b_k + a_1b_{k-1} + \ldots + a_{k-1}b_1 + a_kb_0, \ldots$$

If one desires, he can express this more compactly by

$$c_k = \sum_{i=0}^{k} a_ib_{k-i}, \quad k = 0, 1, 2, \ldots \quad .$$

Now of course this system had identities, is closed, is commutative, and is associative with respect to both operations, and multiplication is distributive over addition. Some of these properties are obvious; others are a little difficult or are somewhat long, at least, to prove. We will omit the proofs here, but let's look at some peculiarities of the system.

The additive identity is $(0, 0, 0, \ldots)$,

the multiplicative identity is $(1, 0, 0, 0, \ldots)$,

the constant polynomial k is $(k, 0, 0, 0, \ldots)$,

the polynomial x is $(0, 1, 0, 0, \ldots)$,

the polynomial kx is $(0, k, 0, 0, 0, \ldots)$,

and the polynomial kx^2 is $(0, 0, k, 0, 0, 0, \ldots)$.

Since $(1, 0, 0, \ldots)$ is the multiplicative identity, we know that

$$(1, 0, 0, 0, \ldots) \text{ X } (a_0, a_1, a_2, \ldots) = (a_0, a_1, a_2, \ldots).$$

What is $(0, 1, 0, 0, \ldots) \times (a_0, a_1, a_2, \ldots)$?

What is $(0, 0, 1, 0, 0, \ldots) \times (a_0, a_1, a_2, \ldots)$?

In general

$$(0, 0, \ldots, 0, \underbrace{1}_{k\text{ th spot}}, 0, 0, \ldots) \times (a_0, a_1, a_2, \ldots)$$

$$= (0, 0, \ldots, 0, \underbrace{a_0}_{k\text{ th spot}}, a_1, a_2, \ldots).$$

One can see that subtraction should be defined by

$$(a_0, a_1, a_2, \ldots) - (b_0, b_1, b_2, \ldots) = (a_0 - b_0, a_1 - b_1, \ldots).$$

It follows immediately that the system is closed with respect to subtraction.

Now let us define division in a natural way:

$$(a_0, a_1, a_2, \ldots) \div (b_0, b_1, b_2, \ldots) = (x_0, x_1, x_2, \ldots)$$

means $$(b_0, b_1, b_2, \ldots) \times (x_0, x_1, x_2, \ldots) = (a_0, a_1, a_2, \ldots)$$

if the indicated polynomial $(x_0, x_1, x_2, \ldots)$ exists.

Let us consider first the case where $b_0 \neq 0$. Thus by the definition of multiplication

$$b_0x_0 = a_0 \text{ and } x_0 = a_0/b_0 ;$$

$$b_0x_1 + b_1x_0 = a_1, \quad b_0x_1 + b_1 \, a_0/b_0 = a_1,$$

$$b_0x_1 = \frac{a_1b_0 - a_0b_1}{b_0} \text{ and } x_1 = \frac{a_1b_0 - a_0b_1}{b_0^2} ;$$

$$b_0x_2 + b_1x_1 + b_2x_0 = a_2, \; b_0x_2 + b_1 \frac{a_1b_0 - a_0b_1}{b_0^2} +$$

$$b_2 \frac{a_0}{b_0} = a_2,$$

$$b_0x_2 = \frac{a_2b_0^2 - a_1b_0b_1 + a_0b_1^2 - a_0b_0b_2}{b_0^2},$$

and $$x_2 = \frac{a_2b_0^2 - a_1b_0b_1 + a_0b_1^2 - a_0b_0b_2}{b_0^3}.$$

Clearly one can continue in this manner to find x_3, x_4, etc. Does this mean that if $b_0 \neq 0$, $(a_0, a_1, a_2, \ldots) \div (b_0, b_1, b_2, \ldots)$ is a polynomial? From our previous knowledge of polynomials we know that this cannot be the case. If there does not exist a number k such that $x_i = 0$ if $i > k$ then the "quotient" $(x_0, x_1, x_2, \ldots)$ is not a polynomial. It is then called a rational function.

We now consider the case $b_0 = 0$.

$$b_0x_0 = a_0$$

has a solution only if $a_0 = 0$. But then x_0 is not determined; it could be any number. We then look at

$$b_0x_1 + b_1x_0 = a_1$$

or $$b_1x_0 = a_1 .$$

Thus $$x_0 = a_1/b_1 \text{ if } b_1 \neq 0.$$

Continuing in this manner one can see that if $(b_0, b_1, b_2, \ldots)$ is not the additive identity, b_k is its first nonzero entry and $a_0 = 0$, $a_1 = 0$ $a_2 = 0, \ldots, a_{k-1} = 0$, then $x_0, x_1, x_2, \ldots$ can be found such that

$$(a_0, a_1, a_2, \ldots) \div (b_0, b_1, b_2, \ldots) = (x_0, x_1, x_2, \ldots).$$

However as in the first case $(x_0, x_1, x_2, \ldots)$ may or may not be a polynomial.

Some high school students should enjoy working with this system. They could discover some of the above properties, if they were given the polynomials with addition and multiplication defined as above. It might prove more interesting if they were't called polynomials until they had worked with them for a while.

ON THE TAKING OF ROOTS— A HISTORICAL NOTE

by KENNETH CUMMINS
Department of Mathematics
Kent State University

It is common knowledge to those who have some background in the history of mathematics that among other of the far-reaching and valuable contributions of the Babylonians was their method of approximating the square root of a rational number. It is remarkable that this very formula or method can be obtained in several ways in our modern classroom and the purpose of this note is to demonstrate this and to suggest some further considerations for students.

The Babylonian method placed a rational number r in the form of $a^2 + h$ where $h < a^2$ and where a and h are (positive) rational or natural numbers and then used $a + h/2a$ as the rational approximation to $\sqrt{a^2 + h}$ for a given h and a given a. For example, by this method $\sqrt{28} = \sqrt{25 + 3} = \sqrt{25 + 3} = \sqrt{5^2 + 3}$ which equals approximately $5 + 3/2.5$ or 5.3 . Hence $\sqrt{28} \doteq 5.3$ by the Babylonian method. Likewise $\sqrt{2} = \sqrt{18/9} = \sqrt{16/9 + 2/9} = \sqrt{(4/3)^2 + 2/9}$ which by the Babylonian method is taken as approximately equal to $4/3 + (2/9 \div 2(4/3)) = 4/3 + 1/12 = 17/12 = 1.417$. A better approximation to $\sqrt{2}$ is obtained if $\sqrt{2}$ is taken as $\sqrt{50/25}$. This yields as an approximation 99/70 or 1.414. The above problems illustrate the method.

Let us now turn to the use of the <u>binomial theorem</u> to approximate roots. Here we would place the number whose square root we desire in the form of $a^2 + h$ and then compute as many terms as we wish in the expansion of $(a^2 + h)^{\frac{1}{2}}$. We have $(a^2 + h)^{\frac{1}{2}} = (a^2)^{\frac{1}{2}} + 1/2\ (a^2)^{-\frac{1}{2}}h + [(1/2)(-1/2) \div 2!](a^2)^{-3/2}h^2 + \ldots = a + h/2a - h^2/8a^3 + \ldots$ and we note that the first two terms of the expansion are exactly the Babylonian expression for square root!

Now let us consider the <u>method of Newton</u>. In this case we seek an approximate zero near $x = a$ for the expression $x^2 - (a^2 + h)$. To do this we set $y = x^2 - (a^2 + h)$ whence $y' = 2x$. At $x = a$ the equation of the tangent line is $y + h = 2a(x-a)$ and the value of x at which the line crosses the x-axis is given by $0 + h = 2ax - 2a^2$ or $x = a + h/2a$. Hence an approximate zero is a +

$h/2a$ which is again the Babylonian approximation if the original number is expressed in the form $a^2 + h$.

In arithmetic we learn that there is also a "guess and divide and average" method for approximating square root. To take the square root of 28 we might "guess" or "take as our first approximation the number 5. $28 \div 5 = 5.60$ to hundredths . In computing the average we have $(5 + 5.60)/2 = 5.30$ as our second approximation. Using this we divide again, $28 \div 5.30 = 5.28$ to hundredths and $(5.30 + 5.28)/2 = 5.29$ becomes a third approximation. Let us now apply this method to $a^2 + h$ and use as a first approximation the number a. Then $(a^2 + h) \div a = a + h/a$ and when we take the average of a and $(a + h/a)$ we obtain $(a + a + h/a)/2$ or $a + h/2a$ which is the Babylonian approximation. Note that we obtained this by averaging our first approximation and our first quotient.

The usual square root algorithm used on $a^2 + h$ yields $a + h/2a$ as the first two terms of the answer as any advanced algebra student can easily confirm.

We have seen that the method of approximating the square root of a number of the form $a^2 + h$ by the Babylonians is the same as that obtained by using the first term or two of the results obtained by the binomial theorem, the method of Newton, the "guess and divide and average" method, and the usual square root algorithm. Not only is this satisfying practice for the student but it increases his already worthy respect for the quality of the contributions of the Babylonians.

But now, armed with our knowledge of algebra and calculus which we do in the high school, we can ask further questions. What three-term expression might the Babylonians have used to approximate the square root of a number of the form $a^2 + h$? Which of the methods above carried to three terms agree with the proposed improvement on the Babylonians? Also, can we devise approximation methods for cube root? Can we devise a "guess and divide and average" method for cube root which will agree with that suggested by the binomial theorem? The student of calculus may wish to treat the binomial expansion of $(a^2 + h)^{\frac{1}{2}}$ as a power series in h and develop a relation between a and h for the series to be convergent. Is it $h < a^2$ as indicated along with the Babylonian method? Herein lie sources of some interesting investigations and fruitful discussions for students -- all coming from the Babylonian method for the approximation of square root.

- Kenneth Cummins

PROFESSIONAL MEMBERSHIP APPLICATION

Date: ________ School: ________ School Address: ________

Position: ☐ teacher, ☐ department head, ☐ supervisor, ☐ student,* ☐ other (specify) ________

Level: ☐ elementary, ☐ junior high school, ☐ high school, ☐ junior college, ☐ college, ☐ other (specify) ________

Other Information: ________

		Amount Paid
State or Province ORGANIZATION: ________	☐ New membership ☐ Renewal membership	
Local ORGANIZATION: ________	☐ New membership ☐ Renewal membership	
OTHER: ________	☐ New membership ☐ Renewal membership	

Name (Please print) ________ Telephone ________

Street Address ________

City ________ State ________ ZIP Code ________

National Council of Teachers of Mathematics	Check one: ☐ New membership ☐ Renewal membership	
	$ 9.00, dues and one journal ☐ *Arithmetic Teacher* or ☐ *Mathematics Teacher*	
	13.00, dues and both journals	
	4.50, student dues and one journal* ☐ *Arithmetic Teacher* or ☐ *Mathematics Teacher*	
	6.50, student dues and both journals*	
	5.00 additional for subscription to *Journal for Research in Mathematics Education* (NCTM members only)	
	.50 additional for individual subscription to *Mathematics Student Journal* (NCTM members only)	
	The membership dues payment includes $4.00 for a subscription to either the *Mathematics Teacher* or the *Arithmetic Teacher* and 25¢ for a subscription to the *Newsletter*. Life membership and institutional subscription information available on request from the Washington office.	

* I certify that I have never taught professionally ________ *(Student signature)* *Enclose One Check for Total Amount Due* →

Fill out, and mail to Dr. Floyd Vest, Mathematics Department, North Texas State University, Denton, Texas 76203.

N O W ! !

TEXAS MATHEMATICS TEACHER
J. William Brown, Editor
Texas Council of Teachers Of Mathematics
Woodrow Wilson High School
100 S. Glasgow Drive
DALLAS, TEXAS 75214